B.1 Theorem. *If S is a subset of $\mathbb{R}^n$, the following are equivalent:*

a. *S is compact.*

b. *If $\mathcal{U}$ is any covering of S by open sets, there is a finite subcollection of $\mathcal{U}$ that still forms a covering of S.*

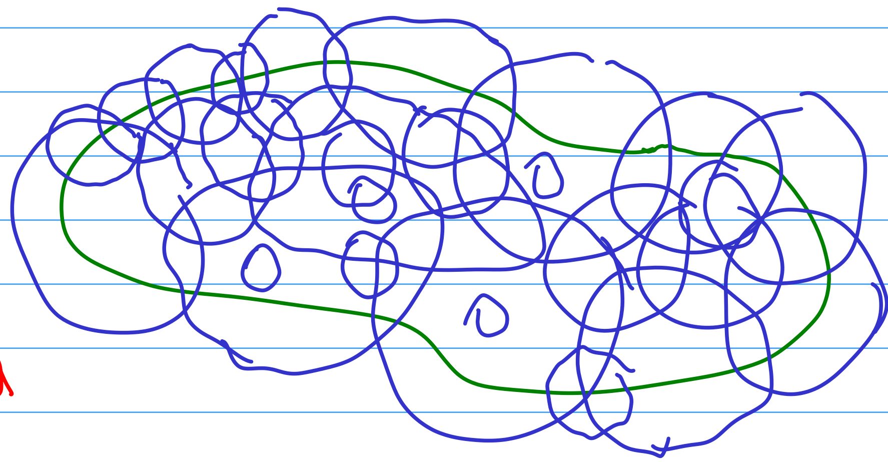
(a) $\Rightarrow$ (b) Suppose S is compact and let $\mathcal{U}$ be a covering of S by open sets.

$$S \subseteq \bigcup_{U \in \mathcal{U}} U = \bigcup \mathcal{U}$$

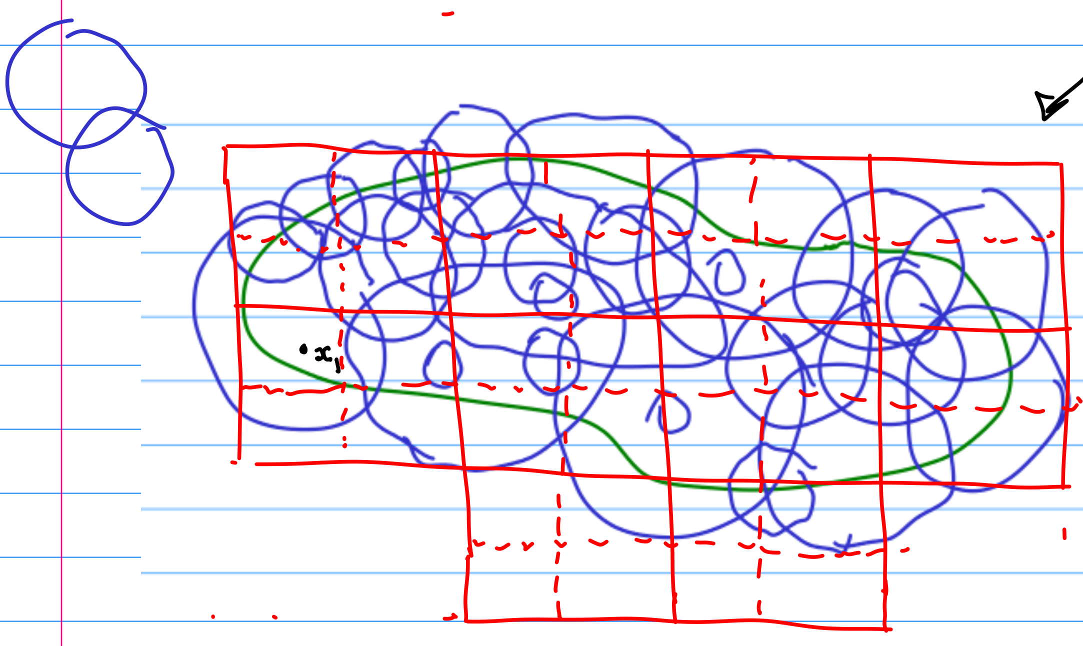
shorthand

Find finite number of sets U_i for $i = 1, \dots, N$ where

$$U_i \in \mathcal{U} \quad \text{and} \quad S \subseteq \bigcup_{i=1}^{N} U_i$$



make a grid



big rectangle contains S but maybe not the cover

Now suppose $\mathfrak{U}$ is a covering of S by open sets. We claim that *there is an integer k such that each box in $\mathcal{B}_k$ that intersects S is included in one of the open sets in $\mathfrak{U}$.* Once we know this, we are done. There are finitely many (in fact, 2^{kn})

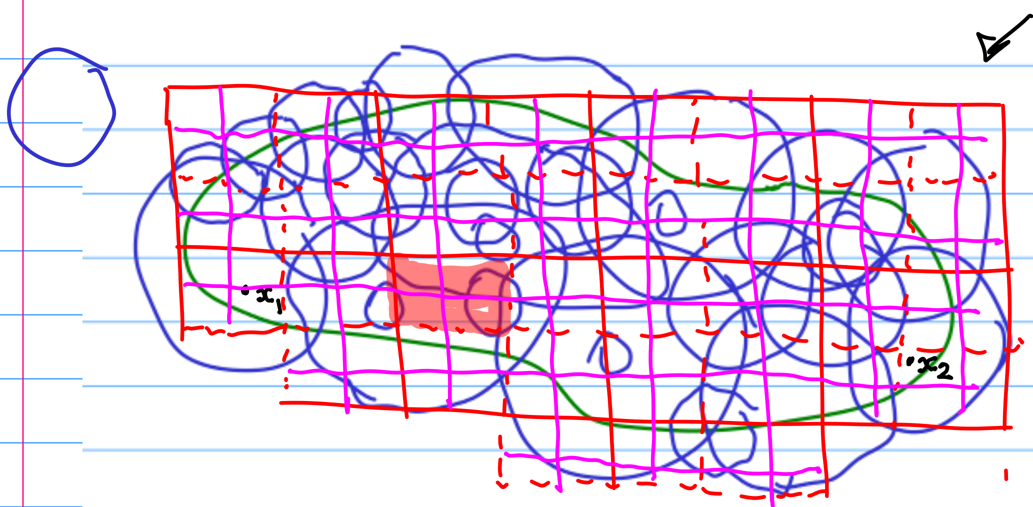
are any of the red boxes that intersect S which are contained entirely in a blue circle?

For each red box that intersect S

is it contained entirely in a blue circle?

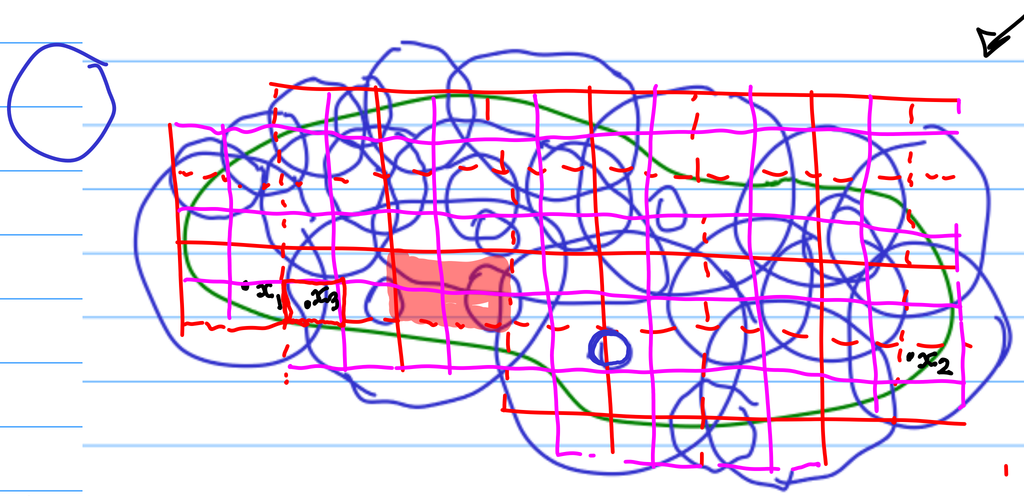
If No then pick a point x_1 in that box which is in S.

Now subdivide the boxes



but may the

Now subdivide again and



but

If at some point each box falls entirely inside a $U \in \mathfrak{U}$

then consider the sets $U_i \in \mathcal{U}$ consisting of those sets.

So for each rectangle choose as U_i one of the sets it's entirely contained in. That's a finite subcollection U_i and since the union of the boxes contain S then $S \subseteq \bigcup_{i=1}^{N} U_i$

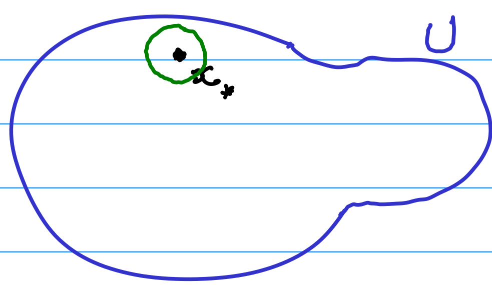
Need to show that eventually the small enough so the above happens.

If not... consider the sequence x_n that we formed while making the grid. $x_n \in S$ and S is compact so the there is a convergent subsequence by Bolzano–Weierstrass Theorem.

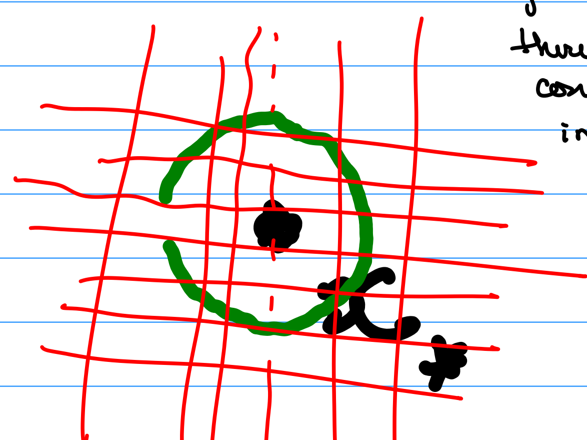
$x_{n_j} \to x_* \in S$ Since $x_* \in S$ then $S \subseteq \bigcup_{U \in \mathcal{U}} U$ implies there is $U \in \mathcal{U}$ so $x_* \in U$.

Since U is open then x_* is some distance ε from the boundary of U

so $B_\varepsilon(x_*) \subseteq U$



since the green ball has radius ε then once the grid gets small enough there is a rectangle containing x_* that in entirely in $B_\varepsilon(x_*)$



let j be large enough that x_{n_j} is so close to x_* that its $|x_{n_j} - x| < \frac{1}{2}\varepsilon$ a

so also the rectangle is contained in U.

Claim this is a contradiction

with $|\mathbf{y} - \mathbf{x}| < \epsilon$ is contained in U. Now pick k large enough so that $|\mathbf{x}_k - \mathbf{x}| < \frac{1}{2}\epsilon$ and also $2^{-k}[\sum_1^n (b_j - a_j)^2]^{1/2} < \frac{1}{2}\epsilon$ The latter condition implies that the distance

Thus there the rectangle that contains x_{n_j} is small enough that it's also in $B_\varepsilon(x_*)$. This contradicts the choice of x_{n_j}

We will use this grid to prove Green's theorem next time...