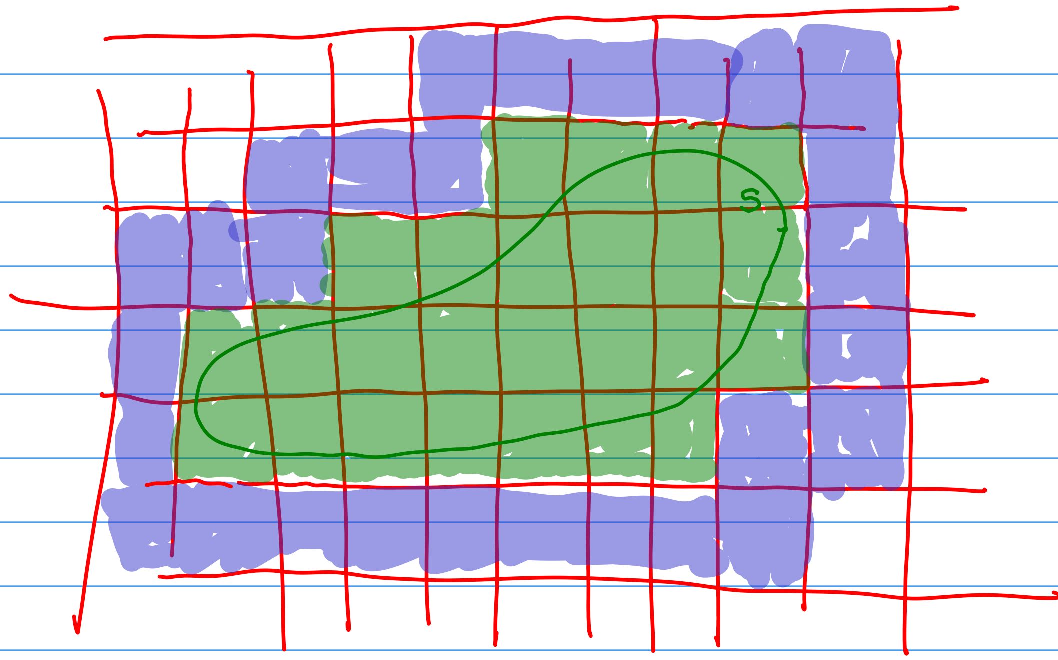


$S \subseteq \bigcup_{j=1}^{N} U_j$ and each rectangle R_m that intersects S is contained entirely in U_j for some $j = j(m)$.

Notation $R_m \subseteq U_{j(m)}$

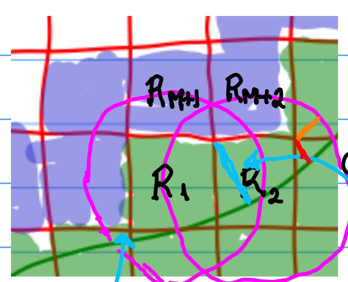
also $S \subseteq \bigcup_{m=1}^{M} R_m$ $\quad R_m = [a_m, b_m] \times [c_m, d_m]$

closed.

R_m for $m = M+1, \dots, N$ are the rectangles that share a boundary with one of the R_m for $m = 1, \dots, M$.

η = minimum width of the rectangles

η



c_2 = distance from R_2 to the boundary of $U_{j(2)}$

c_{M+2} distance from R_{M+2} to the boundary of S.

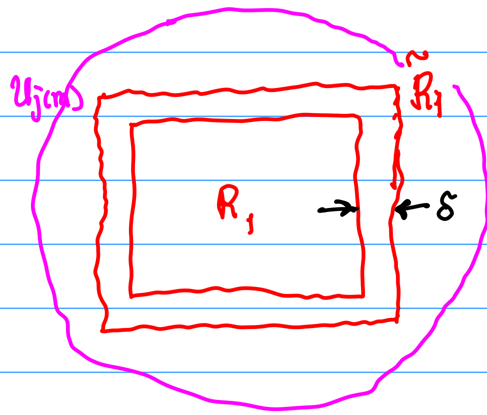
c_{M+1} = distance from R_{M+1} to the boundary of S

c_1 = distance from R_1 to the boundary of $U_{j(1)}$

All these distances are positive

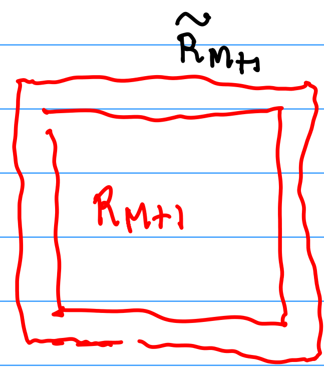
$$\delta = \frac{1}{4}\frac{1}{\sqrt{2}}\min(c_1, c_2, \dots, c_N, \eta)$$

Now consider $\tilde{R}_m = [a_m - \delta, b_m + \delta] \times [c_m - \delta, c_m + \delta]$



Note $\tilde{R}_m \subseteq U_{j(m)}$

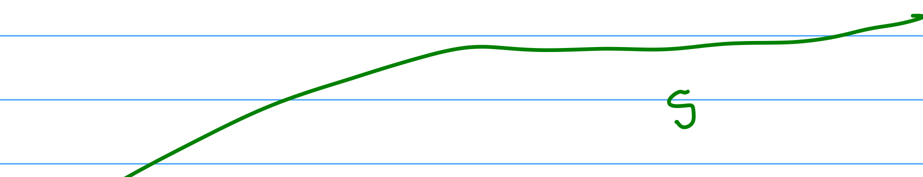
for $m = 1, \dots, M$



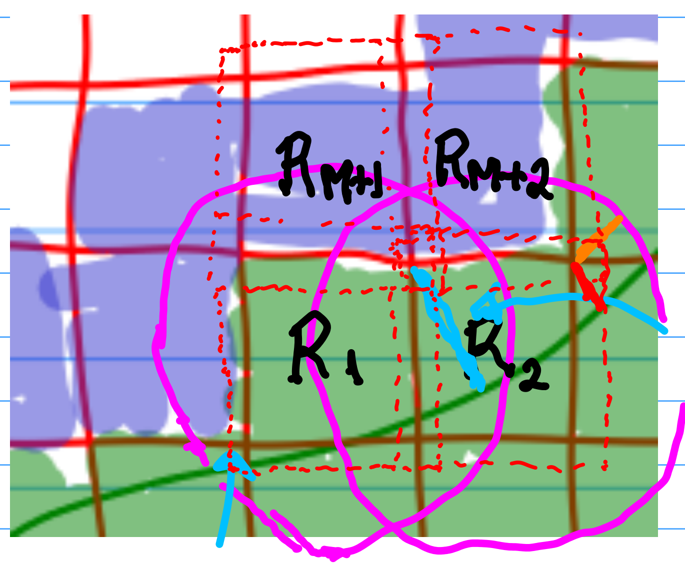
Note since $R_m \cap S \neq \emptyset$

for $m = M+1, \dots, N$

then $\tilde{R}_m \cap S \neq \emptyset$

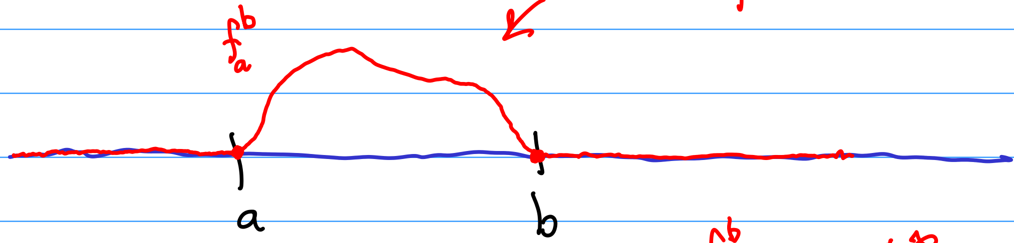


The boxes don't get so big that the boundary of $\tilde{R}_m$ moves past the next adjacent box.... because of η.



Partition of unity.

not only continuous by C^∞



f_a^b is C^∞

$f_a^b(x) = 0$ for $x \le a$ or $x \ge b$

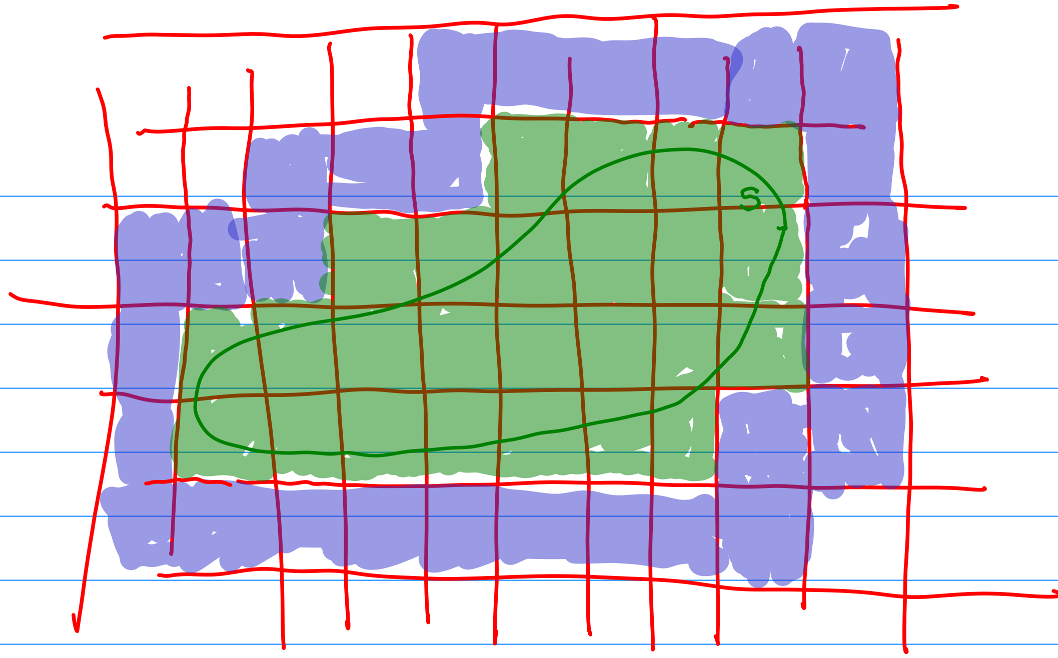
$f_a^b(x) > 0$ for $x \in (a, b)$

$$\tilde{R}_m = [a_m - \delta, b_m + \delta] \times [c_m - \delta, d_m + \delta]$$

$$\psi_m(x,y) = f_{a_m-\delta}^{b_m+\delta}(x)\, f_{c_m-\delta}^{d_m+\delta}(y)$$

product of C^∞ funct. also C^∞.

$$\begin{cases} \psi_m(x,y) > 0 & \text{for } (x,y) \in \tilde{R}_m^{\text{int}} = (a_m-\delta, b_m+\delta) \times (c_m-\delta, d_m+\delta) \\ \psi_m(x,y) = 0 & \text{for } (x,y) \in \partial \tilde{R}_m \text{ or } (x,y) \notin \tilde{R}_m \\ \psi_m \text{ is } C^\infty & \end{cases}$$



$$\sum_{m=1}^{N} \psi_m(x) > 0 \quad \text{on} \quad \bigcup_{m=1}^{N} R_m \quad \text{and subsequently on} \quad \bigcup_{m=1}^{M} \tilde{R}_m$$

Define

$$\varphi_m(x) = \frac{\psi_m(x)}{\sum_{l=1}^{N} \psi_l(x)} \quad \text{for} \quad x \in \bigcup_{m=1}^{M} \tilde{R}_m$$

Note if $x \in S$

Then $\psi_m(x) = 0$ for $m = M+1, M+2, \ldots$ since $\tilde{R}_m \cap S = 0$

$$\sum_{m=1}^{M} \varphi_m(x) = \frac{\sum_{m=1}^{M} \psi_m(x)}{\sum_{l=1}^{N} \psi_l(x)} = \frac{\sum_{m=1}^{N} \psi_m(x)}{\sum_{l=1}^{N} \psi_l(x)} = 1.$$

for $x \in S$.

We have proves...

B.27 Theorem. *Suppose $S \subset \mathbb{R}^n$ is compact and $U_1, \ldots, U_J$ are open sets such that $S \subset \bigcup_1^J U_j$. Then there exists a finite collection $\{\varphi_m\}_1^M$ of C^∞ functions such that*

a. the support of each φ_m is compact and contained in one of the sets U_j, and

b. $\sum_1^M \varphi_m(\mathbf{x}) = 1$ for all $\mathbf{x} \in S$.

Thus.

$$\int_{\partial S} P\,dx + Q\,dy = \int_{\partial S} \sum_{m=1}^{M} P(x,y)\,\varphi_m(x,y)\,dx + Q(x,y)\,\varphi_m(x,y)\,dy$$

$$= \sum_{m=1}^{M} \int_{\partial S} \underbrace{P(x,y)\,\varphi_m(x,y)}_{P_m(x,y)}\,dx + \underbrace{Q(x,y)\,\varphi_m(x,y)}_{Q_m(x,y)}\,dy$$

$$\iint_{S} \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dA = \sum_{m=1}^{M} \iint_{\partial S} \left(\frac{\partial Q_m}{\partial y} - \frac{\partial P_m}{\partial x}\right) dA$$

There is a quiz on Friday. Please know the statement of Theorem 4.41 the change of variables formula for multi-dimensional integrals. It will be a "fill in the blank" quiz, so make sure you remember the statement of the theorem.