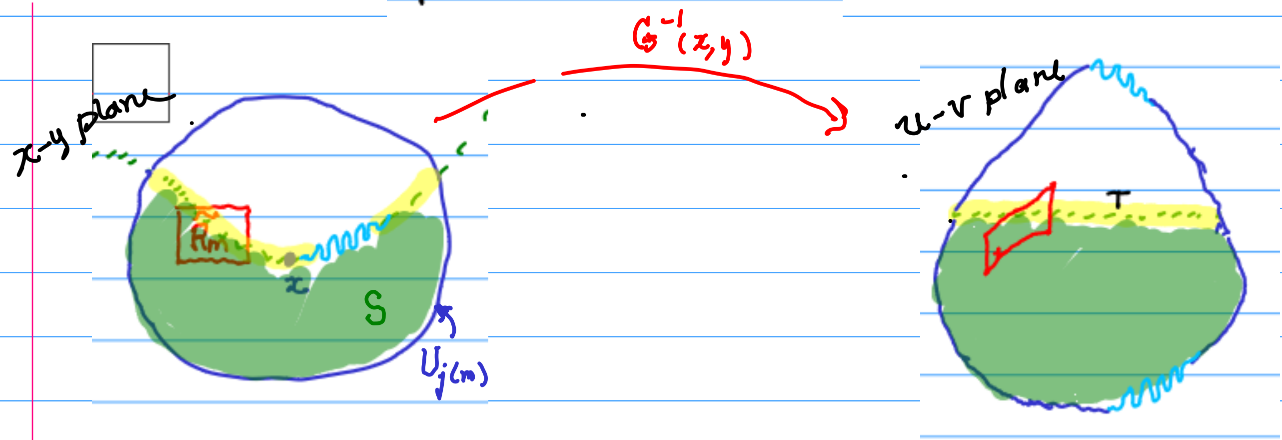
$u = x$ $\qquad v = y - f(x)$

$\boxed{x = u}$ $\qquad y = v + f(x) = v + f(u)$

$G(u,v) = (u, v + f(u))$ $\qquad G^{-1}(x,y) = (x, y - f(x))$

$$\int_{\partial S} P_m\,dx + Q_m\,dy = \int_T (\tilde{P} + \tilde{Q} f'(u))\,du + \tilde{Q}\,dv$$



$$\iint_S \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dA = \iint_{S \cap U_{j(m)}} \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dx\,dy$$

$$= \iint_{G^{-1}(S \cap U_{j(m)})} \left(\frac{\partial \overbrace{Q_m \circ G}^{\tilde{Q}_m}}{\partial x} - \frac{\partial P_m \circ G}{\partial y}\right) |\det DG(u)|\,du\,dv$$

$\tilde{Q} = Q_m \circ G$ $\qquad \tilde{P} = P_m \circ G$

$$\boxed{\frac{\partial \tilde{Q}}{\partial x}} = \frac{\partial \tilde{Q}}{\partial v}\frac{dv}{dx} + \frac{\partial \tilde{Q}}{\partial u}\frac{du}{dx} = \frac{\partial \tilde{Q}}{\partial v}(-f'(x)) + \frac{\partial \tilde{Q}}{\partial u}$$

$$\frac{\partial \tilde{P}}{\partial y} = \frac{\partial \tilde{P}}{\partial v}\frac{dv}{dy} + \frac{\partial \tilde{P}}{\partial u}\frac{du}{dy} = \frac{\partial \tilde{P}}{\partial v} + 0 = \frac{\partial \tilde{P}}{\partial v}$$

$$= \iint_{G^{-1}(S \cap U_{j(m)})} \left(\frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y}\right) |\det DG(u)|\,du\,dv$$

$$= \iint_{G^{-1}(S\cap U_{j(m)})} \left(\frac{\partial \tilde{Q}}{\partial v}(-f'(\underset{u}{x})) + \frac{\partial \tilde{Q}}{\partial u} - \frac{\partial \tilde{P}}{\partial v} \right) |\det DG(u)| \, du\, dv$$

$$= \iint_{G^{-1}(S\cap U_{j(m)})} \left(\frac{\partial \tilde{Q}}{\partial u} - \frac{\partial}{\partial v}\left(\tilde{P} + \tilde{Q} f'(u)\right)\right) |\det DG(u)| \, du\, dv$$

$G(u,v) = (u, v+f(u))$

$$DG(u) = \begin{bmatrix} \frac{\partial G_1}{\partial u} & \frac{\partial G_1}{\partial v} \\ \frac{\partial G_2}{\partial u} & \frac{\partial G_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ f'(u) & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 0 \\ f'(u) & 1 \end{bmatrix} = 1$$

makes sense because G was just a translation.

$$= \iint_{G^{-1}(S\cap U_{j(m)})} \left(\frac{\partial \tilde{Q}}{\partial u} - \frac{\partial}{\partial v}\left(\tilde{P} + \tilde{Q} f'(u)\right)\right) du\, dv$$

$$\int_{\partial S} P_m\, dx + Q_m\, dy = \int_T (\tilde{P} + \tilde{Q} f'(u))\, du + \tilde{Q}\, dv$$

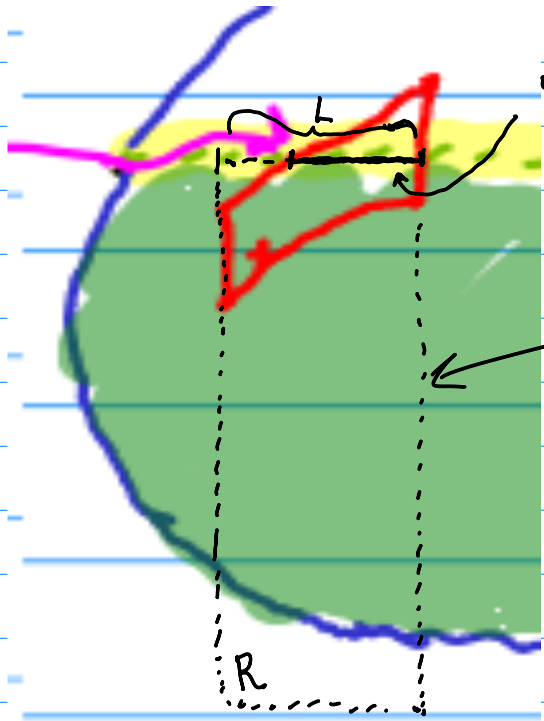
$\| ?$

$$\iint_S \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y} \right) dA = \iint_{G^{-1}(S\cap U_{j(m)})} \left(\frac{\partial \tilde{Q}}{\partial u} - \frac{\partial}{\partial v}\left(\tilde{P} + \tilde{Q} f'(u)\right)\right) du\, dv$$

is anything here a simple region where I can apply the Green's theorem that we've already shown?

NO.

top and bottom of rectangle may have oscillations.



only place on the boundary where P_m and Q_m are non-zero

draw a recctangle with one side the same as L.

Note Q_m and P_m are zero on the other sides of the rectangle...

Also it doesn't have to fit inside $G^{-1}(U_{j(m)})$...

$$\int_T (\tilde{P} + \tilde{Q} f'(u))\,du + \tilde{Q}\,dv = \int_{\partial R} (\tilde{P} + \tilde{Q} f'(u))\,du + \tilde{Q}\,dv$$

since $G^{-1}(S \cap U_{j(m)}) \subseteq R$

R is simple and so the Green's theorem already proves implies that's equal...

$$\iint_{G^{-1}(S \cap U_{j(m)})} \left(\frac{\partial \tilde{Q}}{\partial u} - \frac{\partial}{\partial v}\left(\tilde{P} + \tilde{Q} f'(u)\right)\right) du\,dv = \iint_R \left(\frac{\partial \tilde{Q}}{\partial u} - \frac{\partial}{\partial v}\left(\tilde{P} + \tilde{Q} f'(u)\right)\right) du\,dv$$

Therefore.

$$\int_{\partial s} P_m\,dx + Q_m\,dy = \iint_S \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dA \qquad \text{for every } m\ldots$$

Since by the properties of the partition of unity.

$$\int_{\partial S} P dx + Q dy = \sum_{m=1}^{M} \int_{\partial S} P_m dx + Q_m dy$$

where $P_m(x,y) = P(x,y)\,\phi_m(x,y)$
and $Q_m(x,y) = Q(x,y)\,\phi_m(x,y)$

also

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \sum_{m=1}^{M} \iint_S \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dA$$

Then

5.12 Theorem (Green's Theorem). *Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S. Suppose also that* **F** *is a vector field of class C^1 on $\overline{S}$. Then*

$$\int_{\partial S} \mathbf{F}\cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dA. \tag{5.13}$$

In the more common notation, if we set $\mathbf{F} = (P, Q)$ *and* $\mathbf{x} = (x, y)$,

$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{5.14}$$

5.17 Corollary. *Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S, and let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector to ∂S at $\mathbf{x} \in \partial S$. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on $\overline{S}$. Then*

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dA. \tag{5.18}$$

$$\mathbf{F} \cdot \mathbf{n} = F_1 t_2 - F_2 t_1 = \widetilde{\mathbf{F}} \cdot \mathbf{t}.$$

Also.

$$\int_{\partial S} \nabla f \cdot d\mathbf{x} = \iint_S (\partial_1 \partial_2 f - \partial_2 \partial_1 f)\, dA = \iint_S 0\, dA = 0.$$

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA.$$

Start reading for next time...

Chapter 6

INFINITE SERIES