(Back to single variable analysis)

Infinite series...

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

means the limit of partial sums

$$s_k = \sum_{n=0}^{k} a_n \quad \text{to converge.}$$

Define $\sum_{n=0}^{\infty} a = \lim_{k\to\infty} s_k$. Say the infinite series converges if the limit is finite and exists.

Geometric Series $\sum_{n=0}^{\infty} x^n$ converges iff $|x| < 1$.

$$s_k = \sum_{n=0}^{k} x^n = 1 + x + x^2 + \cdots + x^k$$

$$x s_k = \sum_{n=0}^{k} x^{n+1} = x + x^2 + x^3 + \cdots + x^{k+1}$$

$$(1-x)s_k = 1 - x^{k+1} \qquad \text{assuming } x \neq 1$$

Thus $s_k = \dfrac{1-x^{k+1}}{1-x}$

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1-x^{k+1}}{1-x} = \frac{1}{1-x} \qquad \text{assuming } |x| < 1.$$

so convergent if $|x| < 1$

If $x = 1$ then $s_k = \sum_{n=0}^{k} 1^n = k+1 \to \infty$ as $k \to \infty$

not convergent

$x = -1$ then $s_k = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$

not convergent

for some ξ between 0 and x.

Taylor Theorem

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n + \underbrace{\frac{f^{(k+1)}(\xi)\, x^{k+1}}{(k+1)!}}_{R_k(x)}$$

when does the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ converge?

Need $\boxed{R_k(x) \to 0}$ for some $|x| < c$.

(6.5)
$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

6.6 Theorem. *Let f be a function of class C^∞ on the interval $(-c, c)$, where $0 < c \leq \infty$.*

a. *If there exist constants $a, b > 0$ such that $|f^{(k)}(x)| \leq ab^k k!$ for all $|x| < c$ and $k \geq 0$, then (6.5) holds for $|x| < \min(c, b^{-1})$.*

b. *If there exist constants $A, B > 0$ such that $|f^{(k)}(x)| \leq AB^k$ for all $|x| < c$ and $k \geq 0$, then (6.5) holds for $|x| < c$.*

(a) Suppose $|f^{(k)}(x)| \leq ab^k k!$ for all $|x| < c$ and that $|x| < \min(c, \frac{1}{b})$.

max exists since $|\xi| \leq |x|$ is closed interval and $f^{(k+1)}$ is continuous

$$|R_k(x)| = \left| \frac{f^{(k+1)}(\xi)\, x^{k+1}}{(k+1)!} \right| \leq \max_{|\xi| \leq |x|} \frac{|f^{(k+1)}(\xi)|\, |x|^{k+1}}{(k+1)!}$$

since $|x| < c$ $\qquad$ since $|x| < \frac{1}{b}$ then $b|x| < 1$

$$\leq \frac{a b^{k+1} \cancel{(k+1)!}\, |x|^{k+1}}{\cancel{(k+1)!}} = a (b|x|)^{k+1} \to 0$$

as $k \to \infty$.

Therefore
$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

b. *If there exist constants* $A, B > 0$ *such that* $|f^{(k)}(x)| \le AB^k$ *for all* $|x| < c$ *and* $k \ge 0$, *then* (6.5) *holds for* $|x| < c$.

Suppose $|f^{(k)}(x)| \le AB^k$ for all $|x| < c$ and $|x| < c$.

$$|R_k(x)| = \left| \frac{f^{(k+1)}(\xi)\, x^{k+1}}{(k+1)!} \right| \le \max_{|\xi| \le |x|} \frac{|f^{(k+1)}(\xi)|\,|x|^{k+1}}{(k+1)!}$$

$$\le \frac{AB^{k+1}|x|^{k+1}}{(k+1)!} = \frac{A(Bc)^{k+1}}{(k+1)!} \to 0$$

since the factorial wins against powers.

from 310 argument about sequences

$$\frac{|\alpha|^k}{k!} = \frac{|\alpha|}{1}\frac{|\alpha|}{2}\cdot\frac{|\alpha|}{3}\cdots\frac{|\alpha|}{k}$$

Suppose K is so large that $|\alpha| < K$.

$$\left|\frac{\alpha^k}{k!}\right| \le \frac{|\alpha|}{1}\frac{|\alpha|}{2}\cdot\frac{|\alpha|}{3}\cdots\frac{|\alpha|}{K}\cdot\frac{|\alpha|}{K+1}\cdots\frac{|\alpha|}{k} \quad \text{for } k > K.$$

numerator larger.

denominator smaller

$$\le \underbrace{\frac{|\alpha|}{1}\frac{|\alpha|}{2}\cdot\frac{|\alpha|}{3}\cdots\frac{|\alpha|}{K}}_{M}\,\underbrace{\frac{K}{K+1}\cdots\frac{K}{K+1}}_{k-K \text{ terms}}$$

$K = 3$

$k = 5$

$\frac{|\alpha|}{4}\cdot\frac{|\alpha|}{5}$

$5-3 = 2$ terms

$$= M\left(\frac{K}{K+1}\right)^{-K}\left(\frac{K}{K+1}\right)^{k} \to 0 \text{ as } k \to \infty$$

since $\frac{K}{K+1} < 1$

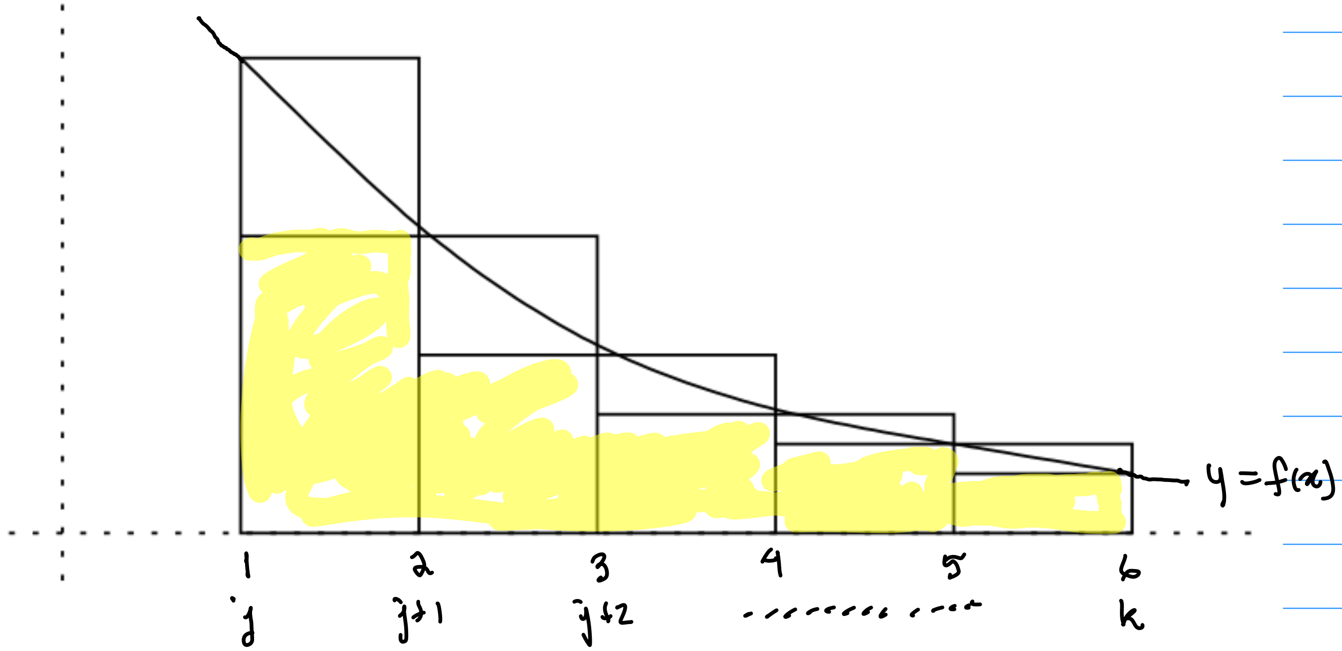
So also in part (b) we have,

$$f(x) = \sum_0^\infty \frac{f^{(n)}(0)}{n!}x^n.$$

6.2 Series with Nonnegative Terms

If $a_n > 0$ for all n,

then $s_k = \sum_{n=0}^{k} a_n$ is a monotone increasing sequence.



$$\sum_{n=2}^{6} f(n) \leq \int_1^6 f(x)\,dx \leq \sum_{n=1}^{5} f(n)$$

In general if $f(x)$ is a decreasing and positive, thus

$$\sum_{n=j+1}^{k} f(n) \leq \int_j^k f(x)\,dx \leq \sum_{n=j}^{k-1} f(n)$$

6.8 Corollary (The Integral Test)**.** *Suppose f is a positive, decreasing function on the half-line $[1, \infty)$. Then the series $\sum_1^\infty f(n)$ converges if and only if the improper integral $\int_1^\infty f(x)\,dx$ converges.*

Proof: By the monotone convergence theorem

Since s_k are increasing, then all we need to do is show the sequence s_k is bounded to ensure it converges.

-line $[1, \infty)$. ! $\int_1^\infty f(x)\,dx$

what is this? improper integral...

By definition

$$\int_1^\infty f(x)\,dx = \lim_{k \to \infty} \int_1^k f(x)\,dx.$$

To show the partial sums are bounded we use.

$$\sum_{n=j+1}^{k} f(n) \le \int_j^k f(x)\,dx$$

Thus

$$s_k = \sum_{n=1}^{k} f(n) = f(1) + \sum_{n=2}^{k} f(n) \le f(1) + \int_1^k f(x)\,dx$$

$$\to f(1) + \underbrace{\int_1^\infty f(x)\,dx}_{\text{finite by assumption}}$$

shows s_k is bounded, monotone and therefore converged.