6.8 Corollary (The Integral Test). *Suppose f is a positive, decreasing function on the half-line $[1, \infty)$. Then the series $\sum_1^\infty f(n)$ converges if and only if the improper integral $\int_1^\infty f(x)\,dx$ converges.*

Example

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $\int_1^\infty \frac{1}{x^p}\,dx$ converges

$$\lim_{K\to\infty} \int_1^K \frac{1}{x^p}\,dx = \lim_{K\to\infty} \begin{cases} \frac{1}{1-p} x^{1-p} \Big|_1^K & \text{for } p \neq 1 \\ \log x \Big|_1^K & \text{for } p = 1 \end{cases}$$

$$= \lim_{K\to\infty} \left\{ \begin{matrix} \frac{1}{1-p}\left(K^{1-p} - 1\right) & \text{for } p \neq 1 \\ \log K & \text{for } p = 1 \end{matrix} \right\} = \begin{cases} 0 & \text{if } p > 1 \\ \infty & \text{if } p < 1 \\ \infty & \text{if } p = 1 \end{cases}$$

Therefore,...

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ otherwise diverges.

note $\int_1^\infty \frac{1}{x^p}$ is not equal $\sum_{n=1}^{\infty} \frac{1}{n^p}$

6.11 Theorem. *Suppose $0 \leq a_n \leq b_n$ for $n \geq 0$. If $\sum_0^\infty b_n$ converges, then so does $\sum_0^\infty a_n$. If $\sum_0^\infty a_n$ diverges, then so does $\sum_0^\infty b_n$.*

Proof. Let $s_k = \sum_0^k a_n$ and $t_k = \sum_0^k b_n$; thus $0 \leq s_k \leq t_k$ for all k. If $\sum_0^\infty b_n$ converges, the numbers t_k form a bounded set; hence so do the numbers s_k, so the sequence $\{s_k\}$ converges by the monotone sequence theorem. This proves the first assertion, to which the second one is logically equivalent. $\square$

6.12 Theorem. *Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers and that a_n/b_n approaches a positive, finite limit as $n \to \infty$. Then the series $\sum_0^\infty a_n$ and $\sum_0^\infty b_n$ are either both convergent or both divergent.*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l > 0 \quad \text{by hypothesis}$$

Given $\varepsilon > 0$ there is N such that $\left|\frac{a_n}{b_n} - l\right| < \varepsilon$ for $n \geq N$.

Choose $\varepsilon = \frac{l}{2}$ then is N such that $\left|\frac{a_n}{b_n} - l\right| < \frac{l}{2}$ for $n \geq N$.

Thus,

$$-\frac{l}{2} < \frac{a_n}{b_n} - l < \frac{l}{2}$$

$$\frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2} \quad \text{for } n \geq N.$$

$$l b_n < 2a_n < 3 l b_n$$

Suppose $k > N$ then

$$\sum_{n=0}^{k} a_n < \frac{1}{2}\sum_{n=0}^{k} 2a_n = \frac{1}{2}\sum_{n=0}^{N-1} 2a_n + \frac{1}{2}\sum_{n=N}^{k} 2a_n$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} 2a_n + \frac{1}{2}\sum_{n=N}^{k} 3l\, b_n$$

If $\sum_{n=0}^{\infty} b_n$ exists then

bounded since finite

bounded since convergent

$$s_k = \sum_{n=0}^{k} a_n < \frac{1}{2}\sum_{n=0}^{N-1} 2a_n + \frac{3l}{2}\sum_{n=0}^{\infty} b_n$$

Since $a_n > 0$ then s_k is monotone increasing and bounded, Therefore $\lim_{k\to\infty} s_k$ exists and $\sum_{n=0}^{\infty} a_n$ converges...

6.13 Theorem (The Ratio Test). *Suppose $\{a_n\}$ is a sequence of positive numbers.*

a. *If $a_{n+1}/a_n < r$ for all sufficiently large n, where $r < 1$, then the series $\sum_0^\infty a_n$ converges. On the other hand, if $a_{n+1}/a_n \geq 1$ for all sufficiently large n, then the series $\sum_0^\infty a_n$ diverges.*

b. *Suppose that $l = \lim_{n\to\infty} a_{n+1}/a_n$ exists. Then the series $\sum_0^\infty a_n$ converges if $l < 1$ and diverges if $l > 1$. No conclusion can be drawn if $l = 1$.*

Let N be so large that $\frac{a_{n+1}}{a_n} < r$ for $n \geq N$

Then $a_{N+1} < r\, a_N$

$$a_{N+2} < r\, a_{N+1} < r^2 a_N$$

$$\vdots$$

$$a_k = a_{N+(k-N)} < r^{k-N} a_N$$

Therefore for $k > N$ we have

$$s_k = \sum_{n=0}^{k} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{k} a_n < \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{k} r^{n-N} a_N$$

only making it smaller so remove it

$$= \sum_{n=0}^{N-1} a_n + a_N \sum_{n=0}^{k-N} r^n = \sum_{n=0}^{N-1} a_n + a_N \frac{1-r^{k-N+1}}{\underbrace{1-r}_{\text{if } r<1 \text{ then } 1-r>0}}$$

$$\leq \underbrace{\sum_{n=0}^{N-1} a_n + a_N \frac{1}{1-r}}_{\text{bounded}}$$

By monotone convergence s_k has a limit.

Read rest of proof from page 289....

6.14 Theorem (The Root Test). *Suppose $\{a_n\}$ is a sequence of positive numbers.*

a. *If $a_n^{1/n} < r$ for all sufficiently large n, where $r < 1$, then the series $\sum_0^\infty a_n$ converges. On the other hand, if $a_n^{1/n} \geq 1$ for all sufficiently large n, then the series $\sum_0^\infty a_n$ diverges.*

b. *Suppose that $l = \lim_{n\to\infty} a_n^{1/n}$ exists. Then the series $\sum_0^\infty a_n$ converges if $l < 1$ and diverges if $l > 1$. No conclusion can be drawn if $l = 1$.*

(a) By hypothesis there is N large enough and $r<1$ so that

$$a_n^{1/n} < r \quad \text{for all} \quad n \geq N$$

Thus $\quad a_n < r^n \quad$ for all $\quad n \geq N$

$$\underset{\nearrow}{s_k} = \sum_{n=0}^{k} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{k} a_n < \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{k} r^n$$

monotone

$$< \sum_{n=0}^{N-1} a_n + \sum_{n=0}^{\infty} r^n = \underbrace{\sum_{n=0}^{N-1} a_n + \frac{1}{1-r}}_{\text{bound}}$$

Thus s_k converges...

A series $\sum_0^\infty a_n$ is called **absolutely convergent** if the series $\sum_0^\infty |a_n|$ converges. For series with nonnegative terms, absolute convergence is the same thing as convergence. For more general series, the basic result is as follows.

6.17 Theorem. *Every absolutely convergent series is convergent.*

$a_n \leq |a_n|$ use comparison test.

$$s_k = \sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} |a_n| = \underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{bound}}$$

no longer monotone

Proof. Suppose $\sum_0^\infty |a_n|$ converges. Let $s_k = \sum_0^k a_n$ and $S_k = \sum_0^k |a_n|$. The sequence $\{S_k\}$ is convergent and hence Cauchy, so given $\epsilon > 0$, there exists an integer K such that

$$|a_{j+1}| + \cdots + |a_k| = S_k - S_j < \epsilon \text{ whenever } k > j \geq K.$$

But then

$$\underline{|s_k - s_j|} = |a_{j+1} + \cdots + a_k| \underset{\text{triangle inequality}}{\leq} \overbrace{|a_{j+1}| + \cdots + |a_j|}^{|S_k - S_j|} < \epsilon \text{ whenever } k > j \geq K,$$

so the sequence $\{s_k\}$ is also Cauchy. By Theorem 1.20, the sequence $\{s_k\}$, and hence the series $\sum a_n$, is convergent. $\square$

Since $\sum |a_k|$ is convergent then S_k is Cauchy.

Since S_k is Cauchy then s_k is Cauchy

Since s_k is Cauchy then $\sum a_k$ is convergent

Check for review sheet over weekend!