Alternating Harmonic Series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1} = \sum_{n=0}^{\infty}(-1)^n a_n$$

where $a_n = \frac{1}{n+1}$.

In general

$a_n > 0$ and decreasing and $\lim_{n\to\infty} a_n = 0$

6.22 Theorem (The Alternating Series Test). *Suppose the sequence $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = 0$. Then the series $\sum_0^\infty (-1)^n a_n$ is convergent. Moreover, if s_k and S denote the kth partial sum and the full sum of this series, we have*

$$s_k > S \text{ for } k \text{ even}, \quad s_k < S \text{ for } k \text{ odd}, \quad \text{and} \quad |s_k - S| < a_{k+1} \text{ for all } k.$$

Since s_{2m} are decreasing the only way $s_{2m} \to S$ is for them all to be greater than S

Since s_{2m+1} are increasing, the only way $s_{2m+1} \to S$ is for them all to be less than S

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots = \underbrace{\left(1-\frac{1}{2}\right)}_{\text{positive}}+\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{\text{positive}}+\underbrace{\left(\frac{1}{5}-\frac{1}{6}\right)}_{\text{positive}}+\cdots$$

$$s_k = \sum_{n=0}^{k}(-1)^n a_n$$

$a_{2m+1} < a_{2m}$

$$s_{2m+1} = \sum_{n=0}^{2m+1}(-1)^n a_n = \underbrace{\sum_{n=0}^{2m-1}(-1)^n a_n}_{s_{2m-1}} + \underbrace{\left(a_{2m} - a_{2m+1}\right)}_{\text{positive}} \geq s_{2m-1}$$

Thus s_{2m+1} is an increasing sequence of m.

$a_{2m+1} > a_{2m+2}$

$$s_{2m+2} = \sum_{n=0}^{2m+2}(-1)^n a_n = \underbrace{\sum_{n=0}^{2m}(-1)^n a_n}_{s_{2m}} \underbrace{- a_{2m+1} + a_{2m+2}}_{\text{negative}} \leq s_{2m}$$

Thus s_{2m} is decreasing sequence of m.

Need to show s_{2m+1} and s_{2m} are bounded sequences to use the monotone convergence theorem.

$$\underset{\text{lower bound}}{s_1} \leq \underset{\substack{\text{increasing}\\\text{sequence.}}}{s_{2m+1}} = \sum_{n=0}^{2m} (-1)^n a_n - a_{2m+1} \leq \underset{\substack{\text{decreasing}\\\text{sequence}}}{s_{2m}} \leq \underset{\text{upper bound.}}{s_0}$$

Then $\lim_{m\to\infty} s_{2m+1} = S_{odd}$ also $\lim_{m\to\infty} s_{2m} = S_{even}$.

Claim $S_{odd} = S_{even}$.

$$|s_{2m+1} - s_{2m}| = |a_{2m+1}| \to 0 \text{ as } m \to \infty$$

Therefore $S_{odd} = S_{even}$.

Since even and odd subsequences have the same limit then s_n converges to that common limit (Math 310).

$$\lim_{k\to\infty} s_k = S = S_{odd} = S_{even}$$

Thus $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = S$ for some S.

Remark this is actually the Taylor series for $\log(1+x)$ expanded about $x=0$ evaluated at $x=1$.

Therefore $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \log 2$. Look at remainder term in Taylor ...

Also note $\sum \left| (-1)^n \frac{1}{n+1} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$

diverges by the (p tests), that is integral

comparison since $\int_1^{\infty} \frac{1}{x} dx = \log x \Big|_1^{\infty} = \infty$.

Given a sequence $a_n \in \mathbb{R}$. i.e. positive and negative.

$$a_n^+ = \max(0, a_n) \qquad a_n^- = \max(-a_n, 0)$$

Thus

$$a_n = a_n^+ - a_n^- \quad \text{and} \quad |a_n| = a_n^+ + a_n^-$$

6.18 Theorem. *If $\sum a_n$ is absolutely convergent, the series $\sum a_n^+$ and $\sum a_n^-$ are both convergent. If $\sum a_n$ is conditionally convergent, the series $\sum a_n^+$ and $\sum a_n^-$ are both divergent.*

Proof. This theorem follows from the following three facts:

i. The convergence of $\sum |a_n|$ implies the convergence of $\sum a_n^+$ and $\sum a_n^-$.

ii. The divergence of $\sum |a_n|$ implies the divergence of at least one of $\sum a_n^+$ and $\sum a_n^-$.

iii. If $\sum a_n$ converges, it cannot happen that one of $\sum a_n^+$ and $\sum a_n^-$ converges while the other one diverges.

Note: conditional convergence means $\sum a_n$ converges but that $\sum |a_n| = \infty$.

the next two even-numbered terms, the next odd-numbered term, and so forth. In general, if σ is any one-to-one mapping from the set of nonnegative integers onto itself, we can form the series $\sum_0^{\infty} a_{\sigma(n)}$, which we call a **rearrangement** of $\sum_0^{\infty} a_n$. (The reasons why we would want to do this are perhaps not so clear right now, but

6.20 Theorem. *Suppose $\sum_0^\infty a_n$ is conditionally convergent. Given any real number S, there is a rearrangement $\sum_0^\infty a_{\sigma(n)}$ that converges to S.*

6.19 Theorem. *If $\sum_0^\infty a_n$ is absolutely convergent with sum S, then every rearrangement $\sum_0^\infty a_{\sigma(n)}$ is also absolutely convergent with sum S.*

proof

$$s_k = \sum_{n=0}^{k} a_n \qquad S_k = \sum_{n=0}^{k} a_{\sigma(n)}$$

Given $K \geq \max\{\sigma(0), \sigma(1), \dots, \sigma(k)\}$

$$\underbrace{\sum_{n=0}^{k} |a_{\sigma(n)}|}_{} \leq \sum_{n=0}^{K} |a_n| \leq \sum_{n=0}^{\infty} |a_n| < \infty$$

by hyp abs conv

Therefore this is a mono increasing sequence of partial sums that's bounds so it has a limit.

Thus, $$\sum_{n=0}^{\infty} |a_{\sigma(n)}| < \infty$$

Let $\varepsilon > 0$. Then there is k large enough so that

$$\sum_{n=k+1}^{\infty} |a_{\sigma(n)}| < \varepsilon.$$

fix that k.

Now choose $K \geq \max\{\sigma(0), \sigma(1), \dots, \sigma(k)\}$ large enough so

$$\sum_{n=K+1}^{\infty} |a_n| < \varepsilon.$$

Note $\{\sigma(0), \sigma(1), \dots, \sigma(k)\} \subseteq \{1, \dots, K\}$

$$\{1, \dots, K\} \setminus \{\sigma(0), \sigma(1), \dots, \sigma(k)\} \subseteq \{\sigma(k+1), \sigma(k+2), \dots\}$$

Since $\sum a_n$ is convergent $S = \sum_{n=0}^{\infty} a_n$

$$\left|\sum_{n=0}^{k} a_{\sigma(n)} - S\right| = \left|\sum_{n=0}^{k} a_{\sigma(n)} - \sum_{n=0}^{K} a_n + \sum_{n=0}^{K} a_n - S\right|$$

$$\le \left|\sum_{n=0}^{k} a_{\sigma(n)} - \sum_{n=0}^{K} a_n\right| + \left|\sum_{n=0}^{K} a_n - S\right|$$

$$\le \sum_{n=k+1}^{\infty} |a_{\sigma(n)}| + \sum_{n=K+1}^{\infty} |a_n| < \varepsilon + \varepsilon = 2\varepsilon$$

Thus $\sum_{n=0}^{\infty} a_{\sigma(n)} = S$.