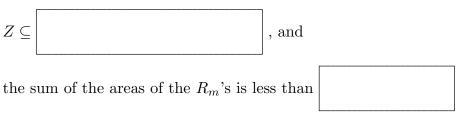
- 1. Fill the missing blanks to make the following definitions correct.
 - (i) A set $Z \subseteq \mathbf{R}^2$ is said to have **zero content** if for every $\varepsilon > 0$ there is a finite collection of rectangles R_1, \ldots, R_M such that



(ii) Let C be a curve described as the range of a one-to-one continuous mapping $g: [a, b] \to \mathbb{R}^n$. Given a partition $P = \{t_0, \ldots, t_J\}$ of [a, b] define



Further let $\mathcal{L} = \{ L_P(C) : P \text{ is a partition of } [a, b] \}$. The curve C is said to be **rectifiable** if the set \mathcal{L} is $\boxed{}$.

2. Let $F: \mathbf{R}^3 \to R^2$ be defined by

$$F(x, y, z) = \begin{bmatrix} 2x + y^2\\\cos(xz) + \sin(yz)\\3y + 2z \end{bmatrix}.$$

Compute the derivative DF and the Jacobian det DF.

- **3.** A set $S \subseteq \mathbf{R}^2$ is Jordan measurable if
 - (A) S is dense in \mathbf{R}^2 and S^c countable.
 - (B) S^c is dense in \mathbf{R}^2 and S countable.
 - (C) S is bounded and its boundary has zero content.
 - (D) $B(r, x) \cap S^c \neq \emptyset$ for every $x \in S$ and r > 0.
 - (E) none of these.
- 4. Let $S \subset \mathbf{R}^n$. The characteristic or indicator function of S is defined by
 - (A) $\chi_S(x) = \begin{cases} 1 & \text{if } x \notin S, \\ 0 & \text{otherwise.} \end{cases}$ (B) $\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$ (C) $\chi_S(x) = \exp(-iSx).$
 - (D) $\chi_S(x) = \int_S f(y)e^{-ixy}dy.$
 - (E) none of these.
- 5. Let $S \subseteq \mathbf{R}^n$ is disconnected if there exists non-empty subsets $S_1, S_2 \subseteq \mathbf{R}^n$ such that
 - (A) $S = S_1 \cap S_2$ where $S_1 \cup \overline{S_2} = \emptyset$ and $\overline{S_1} \cup S_2 = \emptyset$.
 - (B) $S = S_1 \cup S_2$ where $S_1 \cap \overline{S_2} = \emptyset$ and $\overline{S_1} \cap S_2 = \emptyset$.
 - (C) $S = S_1 \cap S_2$ where $S_1 \cup \overline{S_2} \neq \emptyset$ and $\overline{S_1} \cup S_2 \neq \emptyset$.
 - (D) $S = S_1 \cup S_2$ where $S_1 \cap \overline{S_2} \neq \emptyset$ and $\overline{S_1} \cap S_2 \neq \emptyset$.
 - (E) none of these.

6. A set $S \subseteq \mathbf{R}^n$ is called convex if

- (A) whenever $x \in S$ and r > 0 then $B(r, x) \cap S \neq \emptyset$ and $B(r, x) \cap S^c \neq \emptyset$.
- (B) whenever $x \in S$ there exists r > 0 such that $B(r, x) \cap S^c = \emptyset$.
- (C) whenever $a, b \in S$ there is a continuous map $f: [0, 1] \to \mathbb{R}^n$ such that f(0) = a, f(1) = b and $f(t) \in S$ for all $t \in [0, 1]$.
- (D) whenever $a, b \in S$ the line segment from a to b lies in S.
- (E) none of these.

- 7. Please fill in the missing blanks to make the theorems correct.
 - (i) The Change of Variables Theorem for Multiple Integrals: Given open sets U and V in \mathbb{R}^n , let $G: U \to V$ be a one-to-one transformation of class C^1 whose derivative DG(u) is invertible for all $u \in U$. Suppose that $T \subset U$ and $S \subset V$ are measurable sets such that $\overline{T} \subset U$ and G(T) = S. If f is an

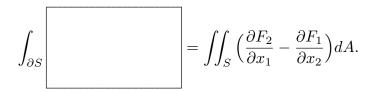
integrable function on S, then $f \circ G$ is

on T, and

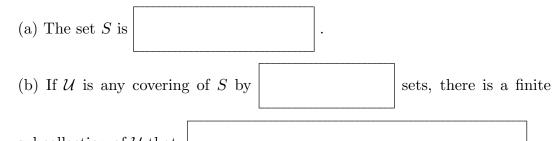
$$\int \cdots \int_S f(x) d^n x = \int \cdots \int_T$$

(ii) Green's Theorem: Suppose S is a

in \mathbb{R}^2 with piecewise smooth boundary ∂S . Suppose also F is a vector field of class C^1 on \overline{S} . Then



(iii) The Heine–Borel Theorem: If S is a subset of \mathbb{R}^n , the following are equivalent:



subcollection of \mathcal{U} that

(iv) The Chain Rule I: Suppose that g(t) is differentiable at t = a and f(x) is differentiable at x = b where b = g(a). Then the composite function $\phi(t) = f(g(t))$ is differentiable at t = a and its derivative is given by



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8. Prove one of the following theorems:

Theorem. Suppose f is a positive, decreasing function on the half-line $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the $\int_{1}^{\infty} f(x) dx$ converges. **Theorem.** The series $\sum_{n=1}^{\infty} n^{-p}$ converges if p > 1 and diverges if $p \le 1$.

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9. Prove one of the following theorems:

Theorem. If f is bounded and monotone on [a, b], then f is integrable on [a, b]. **Theorem.** If f is continuous on [a, b], then f is integrable on [a, b]. Math 311 Final Version A

10. Find $\int_C \sqrt{z} \, ds$ where C is parametrized by $g(t) = (2 \cos t, 2 \sin t, t^2)$ for $0 \le t \le 2\pi$.

11. Sum the geometric series $\sum_{n=3}^{\infty} 5^{-n}$.

12. Give an example of a monotone decreasing sequence $\{a_n\}$ such that $a_n \to 0$ but $\sum_{n=1}^{\infty} a_n$ diverges.

13. Let S be a regular region in R² with piecewise smooth boundary ∂S.
(i) State what it means for S to be a regular region.

(ii) State what it means for S to be x-simple.

(iii) State what it means for S to be y-simple.

14. Recall the Taylor series expanded about a = 0 given by

$$\sin x \sim x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$
 and $\frac{1}{1-x} \sim 1 + x + x^2 + x^3 + x^4 + \cdots$

Find the Taylor polynomial of degree 4 for $f(x,y) = \frac{\sin(x+y)}{1-xy}$ about a = (0,0).

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15. Show that if $f: S \to \mathbb{R}^m$ is uniformly continuous on S and $\{x_k\}$ is a Cauchy sequence in S, then $\{f(x_k)\}$ is also a Cauchy sequence.

16. Give an example of a Cauchy sequence $x_k \in (0, \infty)$ and a continuous function $f: (0, \infty) \to \mathbf{R}$ such that $f(x_k)$ is not Cauchy.

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17. Let $P, Q: \mathbb{R}^2 \to \mathbb{R}$ be continuously differential functions. Let $R = [a, b] \times [c, d]$ and ∂R be the boundary of R oriented in the counter clockwise direction. By definition

$$\int_{\partial R} (Pdx + Qdy) = \int_a^b P(x,c)dx + \int_c^d Q(b,y)dy + \int_b^a P(x,d)dx + \int_d^c Q(a,y)dy.$$

Use the fundamental theorem of calculus and the iterated integral theorem to prove Green's theorem on the rectange R. Namely, that

$$\int_{\partial R} (Pdx + Qdy) = \int_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \, dy.$$