1. Fill the missing blanks to make the following definitions correct.

(i) A set $Z \subseteq \mathbf{R}^2$ is said to have **zero content** if for every $\varepsilon > 0$ there is a finite collection of rectangles $R_1, \ldots, R_M$ such that

$Z \subseteq$ [ ] , and

the sum of the areas of the R_m's is less than [ ] .

(ii) Let C be a curve described as the range of a one-to-one continuous mapping $g\colon [a,b] \to \mathbf{R}^n$. Given a partition $P = \{t_0, \ldots, t_J\}$ of $[a,b]$ define

$$L_P(C) = \boxed{} .$$

Further let $\mathcal{L} = \{\, L_P(C) : P \text{ is a partition of } [a,b] \,\}$. The curve C is said to be **rectifiable** if the set $\mathcal{L}$ is [ ] .

2. Let $F\colon \mathbf{R}^3 \to R^2$ be defined by

$$F(x,y,z) = \begin{bmatrix} 2x + y^2 \\ \cos(xz) + \sin(yz) \\ 3y + 2z \end{bmatrix}.$$

Compute the derivative DF and the Jacobian $\det DF$.

3. A set $S \subseteq \mathbf{R}^2$ is Jordan measurable if

(A) S is dense in $\mathbf{R}^2$ and S^c countable.

(B) S^c is dense in $\mathbf{R}^2$ and S countable.

(C) S is bounded and its boundary has zero content.

(D) $B(r,x) \cap S^c \neq \emptyset$ for every $x \in S$ and $r > 0$.

(E) none of these.

4. Let $S \subset \mathbf{R}^n$. The characteristic or indicator function of S is defined by

(A) $\chi_S(x) = \begin{cases} 1 & \text{if } x \notin S, \\ 0 & \text{otherwise.} \end{cases}$

(B) $\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$

(C) $\chi_S(x) = \exp(-iSx)$.

(D) $\chi_S(x) = \displaystyle\int_S f(y) e^{-ixy} dy$.

(E) none of these.

5. Let $S \subseteq \mathbf{R}^n$ is disconnected if there exists non-empty subsets $S_1, S_2 \subseteq \mathbf{R}^n$ such that

(A) $S = S_1 \cap S_2$ where $S_1 \cup \overline{S_2} = \emptyset$ and $\overline{S_1} \cup S_2 = \emptyset$.

(B) $S = S_1 \cup S_2$ where $S_1 \cap \overline{S_2} = \emptyset$ and $\overline{S_1} \cap S_2 = \emptyset$.

(C) $S = S_1 \cap S_2$ where $S_1 \cup \overline{S_2} \neq \emptyset$ and $\overline{S_1} \cup S_2 \neq \emptyset$.

(D) $S = S_1 \cup S_2$ where $S_1 \cap \overline{S_2} \neq \emptyset$ and $\overline{S_1} \cap S_2 \neq \emptyset$.

(E) none of these.

6. A set $S \subseteq \mathbf{R}^n$ is called convex if

(A) whenever $x \in S$ and $r > 0$ then $B(r,x) \cap S \neq \emptyset$ and $B(r,x) \cap S^c \neq \emptyset$.

(B) whenever $x \in S$ there exists $r > 0$ such that $B(r,x) \cap S^c = \emptyset$.

(C) whenever $a, b \in S$ there is a continuous map $f\colon [0,1] \to \mathbf{R}^n$ such that $f(0) = a$, $f(1) = b$ and $f(t) \in S$ for all $t \in [0,1]$.

(D) whenever $a, b \in S$ the line segment from a to b lies in S.

(E) none of these.

7. Please fill in the missing blanks to make the theorems correct.

(i) The Change of Variables Theorem for Multiple Integrals: Given open sets U and V in $\mathbf{R}^n$, let $G\colon U \to V$ be a one-to-one transformation of class C^1 whose derivative $DG(u)$ is invertible for all $u \in U$. Suppose that $T \subset U$ and $S \subset V$ are measurable sets such that $\overline{T} \subset U$ and $G(T) = S$. If f is an integrable function on S, then $f \circ G$ is ______ on T, and

$$\int \cdots \int_S f(x) d^n x = \int \cdots \int_T \boxed{} .$$

(ii) Green's Theorem: Suppose S is a ______ in $\mathbf{R}^2$ with piecewise smooth boundary ∂S. Suppose also F is a vector field of class C^1 on $\overline{S}$. Then

$$\int_{\partial S} \boxed{} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA.$$

(iii) The Heine–Borel Theorem: If S is a subset of $\mathbf{R}^n$, the following are equivalent:

(a) The set S is ______ .

(b) If $\mathcal{U}$ is any covering of S by ______ sets, there is a finite subcollection of $\mathcal{U}$ that ______ .

(iv) The Chain Rule I: Suppose that $g(t)$ is differentiable at $t = a$ and $f(x)$ is differentiable at $x = b$ where $b = g(a)$. Then the composite function $\phi(t) = f(g(t))$ is differentiable at $t = a$ and its derivative is given by

$$\boxed{} .$$

8. Prove one of the following theorems:

Theorem. Suppose f is a positive, decreasing function on the half-line $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the $\int_1^{\infty} f(x)dx$ converges.

Theorem. The series $\sum_{n=1}^{\infty} n^{-p}$ converges if $p > 1$ and diverges if $p \leq 1$.

9. Prove one of the following theorems:

Theorem. If f is bounded and monotone on $[a, b]$, then f is integrable on $[a, b]$.

Theorem. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

10. Find $\int_C \sqrt{z}\, ds$ where C is parametrized by $g(t) = (2\cos t, 2\sin t, t^2)$ for $0 \leq t \leq 2\pi$.

11. Sum the geometric series $\sum_{n=3}^{\infty} 5^{-n}$.

12. Give an example of a monotone decreasing sequence $\{a_n\}$ such that $a_n \to 0$ but $\sum_{n=1}^{\infty} a_n$ diverges.

13. Let S be a regular region in $\mathbf{R}^2$ with piecewise smooth boundary ∂S.

(i) State what it means for S to be a regular region.

(ii) State what it means for S to be x-simple.

(iii) State what it means for S to be y-simple.

14. Recall the Taylor series expanded about $a = 0$ given by

$$\sin x \sim x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \quad \text{and} \quad \frac{1}{1-x} \sim 1 + x + x^2 + x^3 + x^4 + \cdots.$$

Find the Taylor polynomial of degree 4 for $f(x,y) = \dfrac{\sin(x+y)}{1-xy}$ about $a = (0,0)$.

15. Show that if $f\colon S \to \mathbf{R}^m$ is uniformly continuous on S and $\{x_k\}$ is a Cauchy sequence in S, then $\{f(x_k)\}$ is also a Cauchy sequence.

16. Give an example of a Cauchy sequence $x_k \in (0, \infty)$ and a continuous function $f\colon (0, \infty) \to \mathbf{R}$ such that $f(x_k)$ is not Cauchy.

17. Let $P, Q\colon \mathbf{R}^2 \to \mathbf{R}$ be continuously differential functions. Let $R = [a, b] \times [c, d]$ and ∂R be the boundary of R oriented in the counter clockwise direction. By definition

$$\int_{\partial R} (P dx + Q dy) = \int_a^b P(x, c) dx + \int_c^d Q(b, y) dy + \int_b^a P(x, d) dx + \int_d^c Q(a, y) dy.$$

Use the fundamental theorem of calculus and the iterated integral theorem to prove Green's theorem on the rectange R. Namely, that

$$\int_{\partial R} (P dx + Q dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy.$$