HW2 due Sept 27

Turn in (page 74) 2.3#2abc

Practice (page 41) 1.8#4, 1.8#5, (page 52) 2.1#2, 2.1#9, (page 61-62) 2.2#1abc, 2.2#8, (page 77) 2.5#5

§1.8#4. Show that if $\mathbf{f} : S \to \mathbb{R}^m$ is uniformly continuous on S and $\{\mathbf{x}_k\}$ is a Cauchy sequence in S, then $\{\mathbf{f}(\mathbf{x}_k)\}$ is also a Cauchy sequence. On the other hand, give an example of a Cauchy sequence $\{x_k\}$ in $(0,\infty)$ and a continuous function $f : (0,\infty) \to \mathbb{R}$ (of necessity, *not* uniformly continuous) such that $\{f(x_k)\}$ is not Cauchy.

Let $f: S \to \mathbb{R}^m$ be uniformly continuous on S and $x_k \in S$ be a Cauchy sequence. Claim $f(x_k)$ is also Cauchy.

Let $\varepsilon > 0$. We need to show there is N such that $m,n \geq N$ implies $|f(x_m) - f(x_n)| < \varepsilon$.

Since f is uniformly continuous, there exists $\delta > 0$ such that $x,y \in S$ with $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Since x_k is Cauchy, there exists N such that $m,n \geq N$ implies $|x_m - x_n| < \delta$.

Combining the above together yields that
$$|f(x_m) - f(x_n)| < \varepsilon \quad \text{whenever } m,n \geq N.$$

Therefore $f(x_k)$ is Cauchy.

For the second part consider $f: (0,\infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ and the sequence $x_n = \frac{1}{n}$. Then f is continuous and since $x_n \to 0$ then x_n is Cauchy.

However

$$f(x_{n+1}) - f(x_n) = \frac{1}{x_{n+1}} - \frac{1}{x_n} = n+1-n = 1$$

for all n implies that $f(x_n)$ is not Cauchy.

§1.8 # **5.** Show that if $\mathbf{f} : S \to \mathbb{R}^m$ is uniformly continuous and S is bounded, then $\mathbf{f}(S)$ is bounded.

Suppose $f(S)$ were not bounded. Then there would exist $x_n \in S$ such that $|f(x_n)| \to \infty$.

By choosing a subsequence, if necessary, we may assume that $|f(x_n)| \geq n$ for all $n \in \mathbb{N}$.

Since $x_n \in S$ and S is compact, the Bolzano-Weierstrass theorem implies there is a subsequence $x_{n_k} \to x$ for some $x \in S$.

Let $\varepsilon > 0$ and use the continuity of f at x to find $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$.

Since $x_{n_k} \to x$ there is K large enough so

$$k \geq K \text{ implies } |x_{n_k} - x| < \delta$$

Consequently $k \geq K$ implies $|f(x_{n_k}) - f(x)| < \varepsilon$.

But then

$$n_k \leq |f(x_{n_k})| = |f(x_{n_k}) - f(x)| + |f(x)| < \varepsilon + |f(x)|$$

and the fact that the subsequence $n_k \geq k \to \infty$ as $k \to \infty$ is a contradiction.

Therefore $f(S)$ must be bounded.

§2.1#2. Define the function f by $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Show that f is differentiable at every $x \in \mathbb{R}$, including $x = 0$, but that f' is discontinuous at $x = 0$. (Calculating $f'(x)$ for $x \neq 0$ is easy; to calculate $f'(0)$ you need to go back to the definition of derivative.)

When $x \neq 0$ then

$$f'(x) = 2x \sin(1/x) + x^2 \cos(1/x)(-1/x^2)$$
$$= 2x \sin(1/x) - \cos(1/x)$$

When $x = 0$ then

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin(1/h)$$

Since $|\sin(1/h)| \leq 1$ is bounded and $h \to 0$ we obtain $f'(0) = 0$.

Therefore f is differentiable at every $x \in \mathbb{R}$.

Taking $x_n = \frac{1}{2\pi n}$ yields

$$f'(x_n) = \frac{1}{\pi n} \sin 2\pi n - \cos 2\pi n = -1$$

Taking $y_n = \frac{1}{2\pi(n+1/2)}$ yields

$$f'(y_n) = \frac{1}{\pi(n+1/2)} \sin(2\pi(n+1/2)) - \cos(2\pi(n+1/2)) = 1.$$

If the $\lim_{x\to 0} f'(x)$ were to exist then the limit would be the same along every sequence tending to 0. However x_n and y_n are two sequences tending to 0 and

$$\lim_{n\to\infty} f'(x_n) = -1 \qquad \text{but} \qquad \lim_{n\to\infty} f'(y_n) = 1.$$

Therefore f' is not continuous at 0.

§2.1 **#9.** Define the function f by $f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(0) = 0$.

a. Show that $\lim_{x\to 0} f(x)/x^n = 0$ for all $n > 0$. (You'll find that a simple-minded application of Theorem 2.10 doesn't work. Try setting $y = 1/x^2$ instead.)

Set $y = 1/x^2$ then $y \to \infty$ if and only if $x \to 0$.

Now if $n = 2k$ then

$$\lim_{x\to 0} \frac{f(x)}{x^n} = \lim_{y\to\infty} \frac{e^{-1/x^2}}{x^{2k}} = \lim_{y\to\infty} y^k e^{-y} = \lim_{y\to\infty} \frac{y^k}{e^y}.$$

Since $y^k \to \infty$ and $e^y \to \infty$ as $y \to \infty$ we may apply L'Hôpital's rule in the form of Theorem 2.11 to obtain

$$\lim_{y\to\infty} \frac{y^k}{e^y} = \lim_{y\to\infty} \frac{k y^{k-1}}{e^y}$$

Applying L'Hôpital's rule a total of k times yields

$$\lim_{y\to\infty} \frac{y^k}{e^y} = \lim_{y\to\infty} \frac{k!}{e^y} = 0$$

Therefore

$$\lim_{x\to 0} \frac{f(x)}{x^n} = 0 \quad \text{in the case} \quad n = 2k.$$

If $n = 2k-1$ note that $0 < |x| < 1$ implies $1 \leq \frac{1}{|x|}$. Thus

$$\left|\frac{f(x)}{x^n}\right| \leq \frac{1}{|x|}\left|\frac{f(x)}{x^n}\right| = \left|\frac{f(x)}{x^{2k}}\right| \quad \text{where } k > 0.$$

Since we've already shown that $\left|\frac{f(x)}{x^{2k}}\right| \to 0$ as $x \to 0$

it follows that $\lim_{x\to 0} \frac{f(x)}{x^n} = 0$ as desired.

b. Show that f is differentiable at $x = 0$ and that $f'(0) = 0$.

By definition

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h^{n+1}}$$

Since the last limit was already shown to exist and equal 0, then $f'(0) = 0$.

c. Show by induction on k that for $x \neq 0$, $f^{(k)}(x) = P(1/x)\underbrace{e^{-1/x^2}}_{f(x)}$, where P is a polynomial of degree $3k$.

Suppose $x \neq 0$. Then for $k=1$ we have

$$f'(x) = \frac{d}{dx}\frac{e^{-1/x^2}}{x^n} = \frac{d}{dx}\left(x^{-n}e^{-1/x^2}\right)$$

$$= -nx^{-n-1}e^{-1/x^2} + x^{-n}e^{-1/x^2}\cdot\frac{2}{x^3}$$

$$= \left(-\frac{n}{x}+\frac{2}{x^3}\right)\frac{e^{-1/x^2}}{x^n} = p\left(\frac{1}{x}\right)f(x)$$

where $p(t) = -nt + 2t^3$ is a polynomial of degree 3.

For induction, suppose $f^{(k)}(x) = P_k\left(\frac{1}{x}\right)f(x)$ where P_k is a polynomial of degree $3k$. Claim that

$$f^{(k+1)}(x) = P_{k+1}\left(\frac{1}{x}\right)f(x)$$

where P_{k+1} is a polynomial of degree $3(k+1)$.

Compute as

$$f^{(k+1)}(x) = \frac{d}{dx}f^{(k)}(x) = \frac{d}{dx}P_k\left(\frac{1}{x}\right)f(x) = \frac{d}{dx}P_k\left(\frac{1}{x}\right)\frac{e^{-1/x^2}}{x^n}$$

$$= P_k'\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)\frac{e^{-1/x^2}}{x^n} + P_k\left(\frac{1}{x}\right)\left(-\frac{n}{x}+\frac{2}{x^3}\right)\frac{e^{-1/x^2}}{x^n}$$

$$= \left\{P_k'\left(\frac{1}{x}\right) + P_k\left(\frac{1}{x}\right)\left(-\frac{n}{x}+\frac{2}{x^3}\right)\right\}f(x) = P_{k+1}\left(\frac{1}{x}\right)f(x)$$

where $P_{k+1}(t) = P_k'(t) + P_k(t)\left(-nt+2t^3\right)$.

Since P_k was assumed to be a polynomial of deree $3k$ then P_k' is a polynomial of degree $3k-1$ and

$$P_k(t)(-nt+2t^3) \text{ is a polynomial of degree } 3k+3.$$

It follows that P_{k+1} is a polynomial of degree $3(k+1)$ which completes the induction.

d. Show by induction on k that $f^{(k)}(0)$ exists and equals 0 for all k. (Use the results of (a) and (c) to compute the derivative of $f^{(k-1)}$ at $x=0$ directly from the definition, as in (b).)

Let $k=1$. From part (a) we know that $f'(0)=0$.

For induction suppose $f^{(k-1)}(0)=0$. By definition, part (c) and the induction hypothesis

$$f^{(k)}(0) = \lim_{h\to 0} \frac{f^{(k-1)}(h)-f^{(k-1)}(0)}{h} = \lim_{h\to 0} \frac{P_{k-1}\left(\frac{1}{h}\right)f(h)}{h} = \lim_{h\to 0} Q\left(\frac{1}{h}\right)e^{-1/h^2}$$

where Q is a polynomial of degree $3(k-1)+n+1 = 3k+n+2$. In general writing

$$Q(t) = \sum_{j=0}^{3k+n+2} a_j t^j$$

yields that

$$Q\left(\frac{1}{h}\right)e^{-1/h^2} = \sum_{j=0}^{3k+n+2} a_j \frac{e^{-1/h^2}}{h^j}$$

We know that $\lim_{h\to 0} e^{-1/h^2} = 0$ and $\lim_{h\to 0} \frac{e^{-1/h}}{e^{j}} = 0$ for $j > 0$.

Therefore

$$\lim_{h\to 0} Q\left(\tfrac{1}{h}\right) e^{-1/h^2} = \sum_{j=0}^{3k+n+2} a_j \lim_{h\to 0} \frac{e^{-1/h^2}}{h^j} = \sum_{j=0}^{3k+n+2} a_j \cdot 0 = 0.$$

It follows that $f^{(k)}(0) = 0$ thus completing the induction.

The upshot is that f possesses derivatives of all orders at every point and that $f^{(k)}(0) = 0$ for all k.

§2.2#**1.** For each of the following functions f, (i) compute ∇f, (ii) find the directional derivative of f at the point $(1, -2)$ in the direction $(\frac{3}{5}, \frac{4}{5})$.

a. $f(x, y) = x^2y + \sin \pi xy$.

b. $f(x, y) = e^{4x-y^2}$.

c. $f(x, y) = (x + 2y + 4)/(7x + 3y)$.

Note that $u = \left(\frac{3}{5}, \frac{4}{5}\right)$ is a unit vector, so $\partial_u f(x) = u \cdot \nabla f(x)$.

(a) $\nabla f(x,y) = \left(2xy + \pi y \cos \pi xy,\ x^2 + \pi x \cos \pi xy\right)$

$$\partial_{(\frac{3}{5}, \frac{4}{5})} f(1,-2) = \left(\tfrac{3}{5}, \tfrac{4}{5}\right) \cdot \nabla f(1,-2)$$

$$= \left(\tfrac{3}{5}, \tfrac{4}{5}\right) \cdot \left(-4 - 2\pi \cos 2\pi,\ 1 + \pi \cos 2\pi\right)$$

$$= \left(\tfrac{3}{5}, \tfrac{4}{5}\right) \cdot \left(-4 - 2\pi,\ 1 + \pi\right) = \frac{-12 - 6\pi}{5} + \frac{4 + 4\pi}{5} = \frac{-8 - 2\pi}{5}$$

(b) $f(x,y) = e^{4x-y^2}$

$$\nabla f(x,y) = \left(4e^{4x-y^2}, -2ye^{4x-y^2}\right)$$

$$\partial_{\left(\frac{3}{5},\frac{4}{5}\right)} f(1,-2) = \left(\frac{3}{5},\frac{4}{5}\right)\cdot\left(4e^{4-4}, 4e^{4-4}\right) = \left(\frac{3}{5},\frac{4}{5}\right)(4,4) = \frac{28}{5}$$

(c) $f(x,y) = \dfrac{x+2y+4}{7x+3y}$

$$\nabla f(x,y) = \left(\frac{7x+3y-7(x+2y+4)}{(7x+3y)^2}, \frac{2(7x+3y)-3(x+2y+4)}{(7x+3y)^2}\right)$$

$$= \left(\frac{-11y-28}{(7x+3y)^2}, \frac{11x-12}{(7x+3y)^2}\right)$$

$$\partial_{\left(\frac{3}{5},\frac{4}{5}\right)} f(1,-2) = \left(\frac{3}{5},\frac{4}{5}\right)\cdot(-6,-1) = \frac{-22}{5}$$

§2.2 **#8.** Suppose f is a function defined on an open set $S \subset \mathbb{R}^n$. Show that if the partial derivatives $\partial_j f$ exist and are bounded on S, then f is continuous on S. (Exercise 7 provides an example of a function that satisfies these conditions on $S = \mathbb{R}^2$ but is not everywhere differentiable.)

Suppose the partial derivatives $\partial_j f$ are bounded on S. Then there is a bound M such that $|\partial_j f(x)| \leq M$ for every $x \in S$.

Our argument follows the proof of Theorem 2.19 from the text with changes made as necessary.

For notational convenience consider the case $n=2$.

Let $a \in S$. We wish to show that

$$f(a+h) - f(a) \to 0 \quad \text{as} \quad h \to 0.$$

To do this make the change one variable at a time

$$f(a+h) - f(a) = f(a_1+h_1, a_2+h_2) - f(a_1, a_2+h_2) + f(a_1, a_2+h_2) - f(a_1, a_2)$$

Since S is open we assume h_1 and h_2 are small enough so that the partial derivatives $\partial_j f(x)$ exist whenever $|x-a| \leq |h|$. In this case we can use the one-variable mean value theorem to write

$$f(a_1+h_1, a_2+h_2) - f(a_1, a_2+h_2) = h_1 \partial_1 f(a_1+c_1, a_2+h_2)$$

for some c_1 between 0 and h_1. Similarly

$$f(a_1, a_2+h_2) - f(a_1, a_2) = h_2 \partial_1 f(a_1, a+c_2)$$

for some c_2 between 0 and h_2.

It follows that

$$|f(a+h) - f(a)| \leq |h_1 \partial_1 f(a_1+c_1, a_2+h_2)| + |h_2 \partial_1 f(a_1, a+c_2)|$$

$$\leq |h_1| M + |h_2| M \leq 2|h| M \to 0 \quad \text{as } h \to 0$$

Thus f is continuous for every $a \in S$.

§2.3#2. Find $\partial_x w$ and $\partial_y w$ in terms of the partial derivatives $\partial_1 f$, $\partial_2 f$, and $\partial_3 f$.

a. $w = f(2x - y^2,\, x \sin 3y,\, x^4)$.

b. $w = f(e^{x-3y},\, \log(x^2+1),\, \sqrt{y^4+4})$.

c. $w = \arctan[f(y^2,\, 2x - y,\, -4)]$.

Let $f: \mathbb{R}^3 \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}^3$ be differentiable. Then the chain rule states that

$$\frac{d}{dt} f(g(t)) = \nabla f(g(t)) \cdot g'(t)$$

(a) $w = f(2x-y^2,\, x\sin 3y,\, x^4)$

To compute $\partial_x w$ let $g(x) = (2x-y^2,\, x\sin 3y,\, x^4)$. Then

$$\begin{aligned}\partial_x w &= \nabla f(g(x)) \cdot g'(x) \\ &= \nabla f(2x-y^2, x\sin 3y, x^4) \cdot (2, \sin 3y, 4x^3) \\ &= 2\partial_1 f(2x-y^2, x\sin 3y, x^4) \\ &\quad + \sin 3y\, \partial_2 f(2x-y^2, x\sin 3y, x^4) \\ &\quad + 4x^3 \partial_3 f(2x-y^2, x\sin 3y, x^4).\end{aligned}$$

Similarly

$$\begin{aligned}\partial_y w &= \nabla f(2x-y^2, x\sin 3y, x^4) \cdot (-2y, 3x\cos 3y, 0) \\ &= -2y\, \partial_1 f(2x-y^2, x\sin 3y, x^4) \\ &\quad + 3x\cos 3y\, \partial_2 f(2x-y^2, x\sin 3y, x^4).\end{aligned}$$

b. $w = f(e^{x-3y}, \log(x^2+1), \sqrt{y^4+4})$.

$$\partial_x w = \nabla f\left(e^{x-3y}, \log(x^2+1), \sqrt{y^4+4}\right) \cdot \left(e^{x-3y}, \frac{2x}{x^2+1}, 0\right)$$

$$= e^{x-3f}\, \partial_1 f\left(e^{x-3y}, \log(x^2+1), \sqrt{y^4+4}\right)$$

$$+ \frac{2x}{x^2+1}\, \partial_2 f\left(e^{x-3y}, \log(x^2+1), \sqrt{y^4+4}\right)$$

$$\partial_y w = \nabla f\left(e^{x-3y}, \log(x^2+1), \sqrt{y^4+4}\right) \cdot \left(-3e^{x-3y}, 0, \frac{2y^3}{\sqrt{y^4+4}}\right)$$

$$= -3e^{x-3y}\, \partial_1 f\left(e^{x-3y}, \log(x^2+1), \sqrt{y^4+4}\right)$$

$$\frac{2y^3}{\sqrt{y^4+4}}\, \partial_3 f\left(e^{x-3y}, \log(x^2+1), \sqrt{y^4+4}\right).$$

c. $w = \arctan[f(y^2, 2x-y, -4)]$

$$\partial_x w = (\nabla \arctan \circ f)(y^2, 2x-y, -4) \cdot (0, 2, 0)$$

Now

$$\partial_i \arctan(f(p)) = \frac{1}{f(p)^2+1}\, \partial_i f(p)$$

Therefore

$$\partial_x w = 2\, \frac{\partial_2 f(p)}{f(p)^2+1}\Bigg|_{p=(y^2, 2x-y, -4)}$$

Similarly

$$\partial_y w = (\nabla \arctan\circ f)(y^2, 2x-y, -4) \cdot (2y, -1, 0)$$

Therefore

$$\partial_y w = \left\{ 2y \frac{\partial_1 f(p)}{f(p)^2+1} - \frac{\partial_2 f(p)}{f(p)^2+1} \right\} \Bigg|_{p=(y^2, 2x-y, -4)}$$

§2.5 # **5.** Let $V = \pi r^2 h$ and $S = 2\pi r(r+h)$ (the volume and surface area of a circular cylinder). Compute

$$\left.\frac{\partial V}{\partial h}\right|_r, \quad \left.\frac{\partial V}{\partial h}\right|_S, \quad \left.\frac{\partial V}{\partial S}\right|_r, \quad \left.\frac{\partial S}{\partial V}\right|_r,$$

where the subscript indicates the variable that is being held fixed.

$$\left.\frac{\partial V}{\partial h}\right|_r = \left.\frac{\partial\, \pi r^2 h}{\partial h}\right|_r = \pi r^2$$

To compute $\left.\frac{\partial V}{\partial h}\right|_S$ note that S fixed implies

$$0 = dS = 2\pi r(dr + dh) + 2\pi(r+h)\,dr$$

$$= 2\pi(2r+h)\,dr + 2\pi r\,dh$$

Consequently

$$\left.\frac{dr}{dh}\right|_S = \frac{-2\pi r}{2\pi(2r+h)} = \frac{-r}{2r+h}$$

Thus

$$\left.\frac{\partial V}{\partial h}\right|_S = \left.\frac{\partial V}{\partial h}\right|_r + \left.\frac{\partial V}{\partial r}\right|_h \frac{dr}{dh} = \pi r^2 + 2\pi r h \cdot \frac{-r}{2r+h}$$

$$= \pi r^2 \left(1 - \frac{2h}{2r+h} \right)$$

Now

$$\left.\frac{\partial V}{\partial S}\right|_r = \frac{(\partial V/\partial h)_r}{(\partial S/\partial h)_r} = \frac{\left(\partial \pi r^2 h/\partial h\right)_r}{\left(\frac{\partial\, 2\pi r(r+h)}{\partial h}\right)_r} = \frac{\pi r^2}{2\pi r} = \frac{r}{2}$$

and

$$\left(\frac{\partial S}{\partial V}\right)_r = \frac{1}{\left(\frac{\partial V}{\partial S}\right)_r} = \frac{2}{r}$$