Math 311 Sample Midterm Version A

1. The open ball centered at $a \in \mathbf{R}^n$ of radius $r > 0$ is the set $B(r, a)$ where

(A) $B(r,a) = \{\, x \in \mathbf{R}^n : \|x - a\| \le r \,\}$.

(B) $B(r,a) = \{\, x \in \mathbf{R}^n : \|x - r\| \le a \,\}$.

(C) $B(r,a) = \{\, x \in \mathbf{R}^n : \|x - a\| < r \,\}$.

(D) $B(r,a) = \{\, x \in \mathbf{R}^n : \|x - r\| < a \,\}$.

(E) none of these.

2. Let $S \subseteq \mathbf{R}^n$. The set S^{int} of all interior points of S is given by

(A) $S^{\text{int}} = \{\, x \in S : B(r,x) \subseteq S \text{ for some } r > 0 \,\}$.

(B) $S^{\text{int}} = \{\, x \in S : B(r,x) \supseteq S \text{ for some } r > 0 \,\}$.

(C) $S^{\text{int}} = \{\, x \in S : B(r,x) \subseteq S \text{ for every } r > 0 \,\}$.

(D) $S^{\text{int}} = \{\, x \in S : B(r,x) \supseteq S \text{ for every } r > 0 \,\}$.

(E) none of these.

3. Let $S \subseteq \mathbf{R}^n$ is disconnected if there exists non-empty subsets $S_1, S_2 \subseteq \mathbf{R}^n$ such that

(A) $S = S_1 \cap S_2$ where $S_1 \cup \overline{S_2} = \emptyset$ and $\overline{S_1} \cup S_2 = \emptyset$.

(B) $S = S_1 \cup S_2$ where $S_1 \cap \overline{S_2} = \emptyset$ and $\overline{S_1} \cap S_2 = \emptyset$.

(C) $S = S_1 \cap S_2$ where $S_1 \cup \overline{S_2} \ne \emptyset$ and $\overline{S_1} \cup S_2 \ne \emptyset$.

(D) $S = S_1 \cup S_2$ where $S_1 \cap \overline{S_2} \ne \emptyset$ and $\overline{S_1} \cap S_2 \ne \emptyset$.

(E) none of these.

4. A set $S \subseteq \mathbf{R}^n$ is called convex if

(A) whenever $x \in S$ and $r > 0$ then $B(r,x) \cap S \ne \emptyset$ and $B(r,x) \cap S^c \ne \emptyset$.

(B) whenever $x \in S$ there exists $r > 0$ such that $B(r,x) \cap S^c = \emptyset$.

(C) whenever $a, b \in S$ there is a continuous map $f\colon [0,1] \to \mathbf{R}^n$ such that $f(0) = a$, $f(1) = b$ and $f(t) \in S$ for all $t \in [0,1]$.

(D) whenever $a, b \in S$ the line segment from a to b lies in S.

(E) none of these.

5. A function $f\colon \mathbf{R}^n \to \mathbf{R}$ is said to be differentiable at $a \in \mathbf{R}^n$ if

(A) there exists $c \in \mathbf{R}^n$ such that $\lim_{h\to 0} \dfrac{f(a+h)+f(a)-c\cdot h}{|h|} = 0.$

(B) there exists $c \in \mathbf{R}^n$ such that $\lim_{h\to 0} \dfrac{f(a+h)-f(a)-c\cdot h}{|h|} = 0.$

(C) there exists $c \in \mathbf{R}^n$ such that $\lim_{h\to 0} \dfrac{f(a+h)+f(a)-c\cdot h}{|h|^2} = 0.$

(D) there exists $c \in \mathbf{R}^n$ such that $\lim_{h\to 0} \dfrac{f(a+h)-f(a)-c\cdot h}{|h|^2} = 0.$

(E) none of these.

6. Let $f\colon \mathbf{R}^3 \to \mathbf{R}$ be $f(x,y,z) = x^2+y^3+z^4$ and let α be the multi-index $\alpha = (1,0,3)$.

(i) Find $(\frac{1}{5}, \frac{1}{3}, \frac{1}{2})^\alpha$.

(ii) Find $\partial^\alpha f(3,2,1)$.

7. Please fill in the missing blanks to make the theorem correct.

The Heine–Borel Theorem: If S is a subset of $\mathbf{R}^n$, the following are equivalent:

(a) The set S is [ ] .

(b) If $\mathcal{U}$ is any covering of S by [ ] sets, there is a [ ] subcollection of $\mathcal{U}$ that [ ] .

8. Define a sequence x_k recursively by $x_1 = \sqrt{2}$ and $x_{k+1} = \sqrt{2 + x_k}$. Show by induction that $x_k < 2$ and that $x_k < x_{k+1}$ for all k.

9. State the Bolzano–Weierstrass Theorem in $\mathbf{R}^n$.

10. Please fill in the missing blanks to make the theorem correct.

The Chain Rule I: Suppose that $g(t)$ is differentiable at $t = a$ and $f(x)$ is differentiable at $x = b$ where $b = g(a)$. Then the composite function $\phi(t) = f(g(t))$ is differentiable at $t = a$ and its derivative is given by

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11. Give an example of a Cauchy sequence $x_k \in (0, \infty)$ and a continuous function $f\colon (0, \infty) \to \mathbf{R}$ such that $f(x_k)$ is not Cauchy.

12. Suppose $S \subseteq \mathbf{R}^n$ and $f\colon S \to \mathbf{R}^m$ is continuous at every point of S. If S is compact, then prove that f is uniformly continuous on S.

13. Please fill in the missing blanks to make the theorem correct.

Taylor's Theorem with Integral Remainder: Suppose $f\colon \mathbf{R}^n \to \mathbf{R}$ is of class C^{k+1} on an open convex set $S \subseteq \mathbf{R}^n$. If $a \in S$ and $a + h \in S$, then

$$f(a+h) = \boxed{} + R_k(h)$$

where

$$R_k(h) = (k+1) \sum_{|\alpha|=k+1} \frac{h^\alpha}{\alpha!} \int_0^1 \boxed{} \, dt.$$

14. Recall the Taylor series expanded about $a = 0$ given by

$$\sin x \sim x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \quad \text{and} \quad \frac{1}{1-x} \sim 1 + x + x^2 + x^3 + x^4 + \cdots.$$

Find the Taylor polynomial of degree 4 for $f(x,y) = \dfrac{\sin(x+y)}{1-xy}$ about $a = (0,0)$.