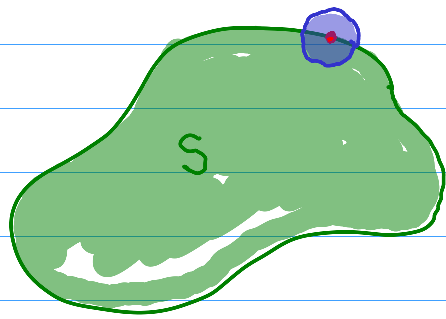
$$B(r, \mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| < r\}.$$

The ball of radius r centered at a.

$$S^{\text{int}} = \{\mathbf{x} \in S : B(r, \mathbf{x}) \subset S \text{ for some } r > 0\}.$$

$$\partial S = \{\mathbf{x} \in \mathbb{R}^n : B(r, \mathbf{x}) \cap S \neq \varnothing \text{ and } B(r, \mathbf{x}) \cap S^c \neq \varnothing \text{ for every } r > 0\}.$$



$$\overline{S} = S \cup \partial S.$$

closure of S.

1.14 Theorem. *Suppose $S \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{x}$ belongs to the closure of S if and only if there is a sequence of points in S that converges to $\mathbf{x}$.*

Proof:

"$\Rightarrow$" Suppose $x \in \overline{S}$.

If $x \in S$ then $x_k = x \in S$ and $x_k \to x$.

If $x \notin S$ then $x \in \partial S$. Let $r_k = \frac{1}{k}$ and consider the balls $B(r_k, x)$. By definition of ∂S these balls intersect S. Thus, $B(r_k, x) \cap S \neq \emptyset$

Consequently there is $x_k \in B(r_k, x) \cap S$ for each k.

Claim $x_k \in S$ and $x_k \to x$. (obvious?)

$$|x - x_k| < \frac{1}{k} \to 0 \quad \text{as} \quad k \to \infty.$$

"$\Leftarrow$" Let $x_k \in S$ with $x_k \to x$ as $k \to \infty$.

Claim $x \in \bar{S}$.

Proof. If $\{\mathbf{x}_k\}$ is a sequence in S that converges to $\mathbf{x}$, then every neighborhood of $\mathbf{x}$ contains elements of S — namely, $\mathbf{x}_k$ where k is sufficiently large — so $\mathbf{x}$ is in the closure of S. Conversely, suppose $\mathbf{x}$ is in the closure of S. If $\mathbf{x}$ is in S itself, let

did I use this?

Case $x \in S$. Let $B(r,x)$ be a neighborhood of x. Then $x_k \to x$ implies there is K such that $x_k \in B(r,x)$ for $k \geq K$.

Since $x_k \in S$ we have $B(r,x) \cap S \neq \emptyset$ for every $r > 0$.

please finish the details at home...

(by contradiction?)

Definition: A subset of $\mathbb{R}^n$ is called **compact** if it is both closed and bounded.

1.21 Theorem (The Bolzano-Weierstrass Theorem). *If S is a subset of $\mathbb{R}^n$, the following are equivalent:*

a. *S is compact.*

b. *Every sequence of points in S has a convergent subsequence whose limit lies in S.*

"$a \Rightarrow b$": Suppose S is compact.

Let $x_k \in S$. Then x_k is bounded since S is bounded and by the Bolzano-Weierstrass theorem from last time it has a convergent subsequence $x_{k_j} \to l$.

Since S is closed then $l \in S$.

"$b \Rightarrow a$": Suppose

Every sequence of points in S has a convergent subsequence whose limit lies in S.

Want to show that S is compact,

For contradiction suppose S is not compact.

Then it is either not bounded or not closed or not either.

If S is not bounded. Then there is $x_k \in S$ such that $x_k \to \infty$ as $k \to \infty$.

But then all subsequences x_{k_j} also $x_{k_j} \to \infty$.

contradicting

Every sequence of points in S has a convergent subsequence whose limit lies in S.

If S is not closed. Then there is $x \in \bar{S} \setminus S$.

Since $x \in \partial S$ then for every $r_k = \frac{1}{k}$ then $B(r_k, x) \cap S \neq \emptyset$

So there is $x_k \in B(r_k, x) \cap S$. Clearly $x_k \to x$ since

$$|x - x_k| < \frac{1}{k} \to 0 \quad \text{as } k \to \infty.$$

Every subsequence x_{k_j} converges to x since the original sequence was convergent contradicting

Every sequence of points in S has a convergent subsequence whose limit lies in S.

1.22 Theorem. *Continuous functions map compact sets to compact sets. That is, suppose that S is a compact subset of $\mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$ is continuous at every point of S. Then the set*

$$\mathbf{f}(S) = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in S\}$$

is also compact.

Proof: Suppose S is compact. Claim $f(S)$ is compact.

Let $y_k \in f(S)$ need to show y_k has a conv. subseq. and the limit is in $f(S)$.

Let $x_k \in S$ such that $y_k = f(x_k)$

Since S is compact there is a subseq. $x_{k_j} \to x \in S$.

Then

by continuity of f

$$\lim_{j\to\infty} y_{k_j} = \lim_{j\to\infty} f(x_{k_j}) = f\left(\lim_{j\to\infty} x_{k_j}\right) = f(x) \in f(S)$$

1.23 Corollary (The Extreme Value Theorem)**.** *Suppose $S \subset \mathbb{R}^n$ is compact and $f : S \to \mathbb{R}$ is continuous. Then f has an absolute minimum value and an absolute maximum value on S; that is, there exist points $\mathbf{a}, \mathbf{b} \in S$ such that $f(\mathbf{a}) \le f(\mathbf{x}) \le f(\mathbf{b})$ for all $\mathbf{x} \in S$.*

Proof. By Theorem 1.22, the set $f(S)$ is a compact subset of $\mathbb{R}$. Thus, it is *bounded,* so $\inf f(S)$ and $\sup f(S)$ exist, and *closed,* so $\inf f(S)$ and $\sup f(S)$ actually belong to $f(S)$. But this says precisely that the set of values of f on S has a smallest and a largest element, as desired. □

1.24 Theorem (The Heine-Borel Theorem)**.** *If S is a subset of $\mathbb{R}^n$, the following are equivalent:*

a. *S is compact.*
b. *If $\mathcal{U}$ is any covering of S by open sets, there is a finite subcollection of $\mathcal{U}$ that still forms a covering of S. (In brief: Every open covering of S has a finite subcovering.)*

please read for Wednesday

Proof. The proof is given in Appendix B.1 (Theorem B.1). □