1.14 Theorem. *Suppose $S \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{x}$ belongs to the closure of S if and only if there is a sequence of points in S that converges to $\mathbf{x}$.*

"$\Rightarrow$" done

"$\Leftarrow$" Let $x_k \in S$ with $x_k \to x$ as $k \to \infty$

Claim $x \in \bar{S}$.

Suppose for contradiction that $x \notin \bar{S}$.

Thus $x \in \bar{S}^c$.

Since $\bar{S}$ is closed then $\bar{S}^c$ is open.

Thus every point in $\bar{S}^c$ is an interior point so there is $r>0$ such that $B(r,x) \subseteq \bar{S}^c$.

Since $x_k \to x$ there is K so large $k \geq K$ implies $x_k \in B(r,x)$,

$x_k \in S$ implies $B(r,x) \cap S \neq \emptyset$.

$B(r,x) \subseteq \bar{S}^c$ so $B(r,x) \cap S \subseteq \bar{S}^c \cap S \subseteq \bar{S}^c \cap \bar{S} = \emptyset$

which is a contradiction

Therefore $x \in \bar{S}$.

B.1 The Heine-Borel Theorem

B.1 Theorem. *If S is a subset of $\mathbb{R}^n$, the following are equivalent:*

a. S is compact.

b. If $\mathfrak{U}$ is any covering of S by open sets, there is a finite subcollection of $\mathfrak{U}$ that still forms a covering of S.

Proof. If S is not compact, by the Bolzano-Weierstrass theorem there is a sequence $\{\mathbf{x}_k\}$ in S, no subsequence of which converges to any point of S. This means that for each $\mathbf{x} \in S$ there is an open ball $D_{\mathbf{x}}$ centered at $\mathbf{x}$ that contains $\mathbf{x}_k$ for at most

please please try to read this proof for next time...

Definition:

is as follows: A set $S \subset \mathbb{R}^n$ is **disconnected** if it is the union of two nonempty subsets S_1 and S_2, neither of which intersects the closure of the other one; in this

Definition a set $S \subseteq \mathbb{R}^n$ is connected if it is not **disconnected**.

1.25 Theorem. *The connected subsets of $\mathbb{R}$ are precisely the intervals (open, half-open, or closed; bounded or unbounded).*

harder

"$\Rightarrow$" If $S \subseteq \mathbb{R}$ is an interval then it's connected.

"$\Leftarrow$" If $S \subseteq \mathbb{R}$ is connected then it's an interval.

easier

Do the easier one first

contrapositive

If $S \subseteq \mathbb{R}$ is not an interval then it's ~~not connected~~ disconnected

Need to find $S_1, S_2 \subseteq \mathbb{R}$ with $S_1 \neq \emptyset$, $S_2 \neq \emptyset$,
$S = S_1 \cup S_2$ and $S_1 \cap \overline{S_2} = \emptyset$ and $\overline{S_1} \cap S_2 = \emptyset$.

Proof. If $S \subset \mathbb{R}$ is not an interval, there exist $a, b \in S$ and $c \notin S$ such that $a < c < b$. Let $S_1 = S \cap (-\infty, c)$ and $S_2 = S \cap (c, \infty)$. Then $S = S_1 \cup S_2$ (since $c \notin S$), and S_1 and S_2 are nonempty since $a \in S_1$ and $b \in S_2$. The closures of S_1 and S_2 are contained in $(-\infty, c]$ and $[c, \infty)$, so the only point where they can intersect is c, which is not in either S_1 or S_2. Thus S is disconnected.

"$\Rightarrow$" If $S \subseteq \mathbb{R}$ is an interval then it's connected.

Case $S = [a, b]$. I.e. case S is a closed interval.

Claim that S is connected. For contradiction suppose it is disconnected.

There exists $S_1, S_2 \subseteq \mathbb{R}$ with $S_1 \neq \emptyset$, $S_2 \neq \emptyset$,
$S = S_1 \cup S_2$ and $S_1 \cap \overline{S_2} = \emptyset$ and $\overline{S_1} \cap S_2 = \emptyset$.

By relabeling, if necessary, assume $b \in S_2$.

Let $c = \sup S_1$. Then $c \in \overline{S_1}$ and consequently $c \notin S_2$.

Since S is closed and $S_1 \subseteq S$ then $\overline{S_1} \subseteq \overline{S} = S$.

So $c \in S = S_1 \cup S_2$. Since $c \notin S_2$ then $c \in S_1$.

Note also $c \neq b$ and $(c, b] \cap S_1 = \emptyset$.

Claim $c \in \overline{S_2}$. Since $S = [a,b] = S_1 \cup S_2$.

then $(c, b] \subseteq S \setminus S_1 = S_2$.

But then $\overline{(c, b]} \subseteq \overline{S_2}$

$[c, b] \subseteq \overline{S_2}$ so $c \in \overline{S_2}$

Thus $c \in S_1 \cap \overline{S_2}$ contradicting that $S_1 \cap \overline{S_2} = \emptyset$.