From last year...

Theorem 5.4 (Generalized Mean Value Theorem) Let f and g be continuous functions on $[a, b]$ that are differentiable on (a, b). Then there is a c in (a, b) such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

The current book...

2.9 Theorem (Mean Value Theorem II). *Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that* ← additional hypothesis.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

It's easy to add the additional hypothesis to Theorem 5.4 and get Theorem 2.9. Can you easily prove 5.4 from 2.9?

Note if I take $g(x) = x$ in $g'(x) = 1$.

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

Then $\underbrace{f(b) - f(a) = (b-a)f'(c)}_{\text{usual mean value theorem from math 181.}}$ for some $c \in (a,b)$

2.19 Theorem. *Let f be a function defined on an open set in $\mathbb{R}^n$ that contains the point $\mathbf{a}$. Suppose that the partial derivatives $\partial_j f$ all exist on some neighborhood of $\mathbf{a}$ and that they are continuous at $\mathbf{a}$. Then f is differentiable at $\mathbf{a}$.*

Example:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{for } (x,y) \neq 0 \\ 0 & \text{for } (x,y) = 0. \end{cases}$$

$$\partial_1 f(x,y) = \begin{cases} \dfrac{y(x^2+y^2) - xy\,2x}{(x^2+y^2)^2} = \dfrac{y^3 - x^2 y}{(x^2+y^2)^2} & \text{for } (x,y) \neq 0 \\ \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \dfrac{0-0}{h} = 0 & \text{for } (x,y) = 0 \end{cases}$$

1 means with respect to the first argument

$$\partial_2 f(x,y) = \begin{cases} \dfrac{x(x^2+y^2) - xy\,2y}{(x^2+y^2)^2} = \dfrac{x^3 - x y^2}{(x^2+y^2)^2} & \text{for } (x,y) \neq 0 \\ \lim_{h\to 0} \dfrac{f(0,0+h) - f(0,0)}{h} = \dfrac{0-0}{h} = 0 & \text{for } (x,y) = 0 \end{cases}$$

2 means with respect to the second argument

This is a situation where the partial derivatives $\partial_1 f$ and $\partial_2 f$ exist for all values of (x,y).

From last time

Proof. Multiplying (2.15) through by $|\mathbf{h}|$, we see that $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h} \to 0$ as $\mathbf{h} \to \mathbf{0}$. Since $\nabla f(\mathbf{a}) \cdot \mathbf{h}$ clearly vanishes as $\mathbf{h}$ does, we have $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \to 0$ as $\mathbf{h} \to \mathbf{0}$, which says precisely that f is continuous at $\mathbf{a}$. □

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h} \to 0 \quad \text{as } h \to 0$$

Thus $\quad f(a+h) - f(a) \to 0 \quad$ since $\nabla f(a) \cdot h \to 0$

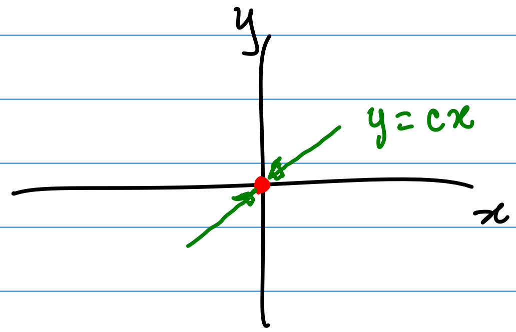
Thus f should be continuous if it's differentiable.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{for } (x,y) \neq 0 \\ 0 & \text{for } (x,y) = 0. \end{cases}$$

Is f continuous at $(x,y)=0$.

Is $\lim\limits_{(x,y)\to 0} f(x,y) = 0$?

if this is true for any way that $(x,y)\to 0$
then it's true for a specific way for $(x,y)\to 0$
For example, let $y=cx$ and $x\to 0$



approach $(0,0)$ on a line.

$$\lim_{x\to 0} f(x,cx) = \lim_{x\to 0} \frac{\cancel{x}\, c\cancel{x}}{\cancel{x^2}+(c\cancel{x})^2} = \frac{c}{1+c^2} \neq 0 \text{ for } c\neq 0$$

and depending on c in general,

So f is not cont. at $(0,0)$ which mean that ∇f does not exist even though $\partial_1 f$ and $\partial_2 f$ exist at $(0,0)$.

So when does the existence of ∂f and $\partial_2 f$ imply f is differentiable?

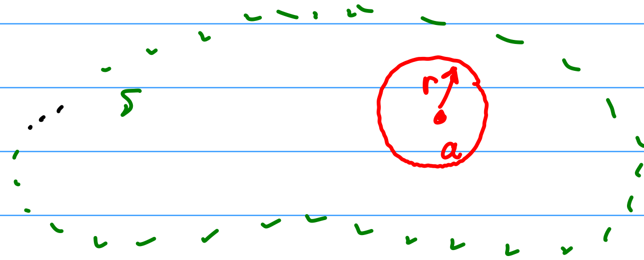
2.19 Theorem. *Let f be a function defined on an open set in $\mathbb{R}^n$ that contains the point* **a**. *Suppose that the partial derivatives $\partial_j f$ all exist on some neighborhood of* **a** *and that they are continuous at* **a**. *Then f is differentiable at* **a**.

f differentiable means there is a c such that

$$\frac{E(h)}{|h|} \to 0 \text{ as } h\to 0 \text{ where } E(h) = \underbrace{f(a+h)}_{\text{function}} - \underbrace{f(a) - c\cdot h}_{\text{linear approx.}}$$

Let $c = (\partial_1 f(a), \partial_2 f(a), \dots, \partial_n f(a))$. Claim $\frac{E(h)}{|h|} \to 0$.

Let $S \subseteq \mathbb{R}^n$ that's open and $a \in S$. Then there is $r>0$ such that $B(r,a) \subseteq S$.



Make r smaller if needed so that $\partial_1 f, \partial_2 f, \dots, \partial_n f$ are exist on $B(r,a)$ and are continuous..

recall
$f(b)-f(a) = (b-a)f'(c)$ for some $c \in (a,b)$

Now assume $n=2$ for notational simplicity

$f(a+h)-f(a) = f(a_1+h_1, a_2+h_2) - f(a_1,a_2)$ $a=(a_1,a_2)$, $h=(h_1,h_2)$

$$= \underbrace{f(a_1+h_1, a_2+h_2) - f(a_1, a_2+h_2)}_{\text{mean value theorem on the first argument}} + \underbrace{f(a_1, a_2+h_2) - f(a_1,a_2)}_{\text{mean value theorem on the second argument}}$$

$$= h_1 \partial_1 f(c_1, a_2+h_2) + h_2 \partial_2 f(a_1, c_2)$$

c_2 depends of h_2

note that c_1 depends on h_1 and h_2.

where c_1 is between a_1 and a_1+h_1 and c_2 is between a_2 and a_2+h_2

$$\frac{E(h)}{|h|} = \frac{h_1 \partial_1 f(c_1, a_2+h_2) + h_2 \partial_2 f(a_1,c_2) - c\cdot h}{|h|}$$

recall
$c = (\partial_1 f(a), \partial_2 f(a))$

$$\frac{E(h)}{|h|} = \frac{h_1 \partial_1 f(c_1, a_2+h_2) - h_1 \partial_1 f(a) + h_2 \partial_2 f(a_1, c_2) - h_2 \partial_2 f(a)}{|h|}$$

$$= \underbrace{\frac{h_1}{|h|}}_{\text{bounded}} \Big(\underbrace{\partial_1 f(c_1, a_2+h_2) - \partial_1 f(a)}_{\text{tends to } 0} \Big) + \underbrace{\frac{h_2}{|h|}}_{\text{bounded.}} \Big(\underbrace{\partial_2 f(a_1, c_2) - \partial_2 f(a)}_{\text{tends to zero.}} \Big) \rightarrow 0 \quad \underbrace{\text{as } h \to 0}$$

$|h| = \sqrt{h_1^2 + h_2^2} \geq |h_1|$

$|h| = \sqrt{h_1^2 + h_2^2} \geq |h_2|$

since c_1 is between a_1 and a_1+h_1 then $c_1 \to a_1$ as $|h| \to 0$

since c_2 is between a_2 and a_2+h_2 then $c_2 \to a_2$ as $|h| \to 0$

also $a_2+h_2 \to a_2$ as $|h| \to 0$

Thus $(c_1, a_2+h_2) \to a$ and $(a_1, c_2) \to a$ as $|h| \to 0$

so $\partial_1 f(c_1, a_2+h_2) - \partial_1 f(a) \to 0$ and $\partial_2 f(a_1, c_2) - \partial_2 f(a) \to 0$

since $\partial_1 f$ and $\partial_2 f$ are continuous at a.