2.42 Theorem. *Suppose that f is differentiable on an open connected set S and $\nabla f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in S$. Then f is constant on S.*

Proof: Idea Fix $a \in S$ and define

$$\begin{cases} S_1 = \{x \in S : f(x) = f(a)\} \\ S_2 = S \setminus S_1 = \{x \in S : f(x) \neq f(a)\} \end{cases}$$

$\uparrow$ hold.

We know that S is connected so S_1, S_2 better not be a disconnection of S.

$S = S_1 \cup S_2$
$S_1 \cap S_2 = \emptyset$
try to show $S_2 = \emptyset$.
If $S_1 \cap \bar{S}_2 = \emptyset$
and $\bar{S}_1 \cap S_2 = \emptyset$
then S_1, S_2 would be a disconnection unless $S_2 = \emptyset$.

means $f(x) \neq f(a)$

Claim S_2 is open: Let $x \in S_2$. Since $S_2 \subseteq S$ then there is $\rho > 0$ such that $B(\rho, x) \subseteq S$.

Since f is continuous Given $\varepsilon = \dfrac{|f(x) - f(a)|}{2} > 0$ there is $\delta > 0$ such that $|x - y| < \delta$ and $y \in S$ implies that $|f(x) - f(y)| < \varepsilon$.

Consequently

$$|f(y) - f(a)| \geq |f(x) - f(a)| - |f(x) - f(y)| \geq 2\varepsilon - \varepsilon = \varepsilon > 0$$

so $y \in S_2$. $\quad r = \min(\delta, \rho)$

then $B(r, x) \subseteq S$ and every $y \in B(r, x)$ is also in S_2

So $B(r, x) \subseteq S_2$

This means S_2 is open... Then $\bar{S}_1 \cap S_2 = \emptyset$. $\boxed{\checkmark}$

Next show $S_1 \cap \bar{S}_2 = \emptyset$. ...

Generalize this

2.41 Corollary. *Suppose f is differentiable on an open convex set S and $\nabla f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in S$. Then f is constant on S.*

Claim S_1 is open. Recall $S_1 = \{x \in S : f(x) = f(a)\}$

Let $x \in S_1$. Since $S_1 \subseteq S$ and S is open, then there is $\rho > 0$ such that $B(\rho, x) \subseteq S$.

Since $B(\rho, x)$ is convex then Corollary 2.41 implies that f is constant on $B(\rho, x)$.

Thus $y \in B(\rho, x)$ implies $f(y) = f(x) = f(a)$ Thus $B(\rho, x) \subseteq S_1$.

This means S_1 is open and so $S_1 \cap \bar{S}_2 = \emptyset$.

Since $S_2 \neq \emptyset$ would contradict the assumption S is connected then we must have that $S_2 = \emptyset$ and so $S_1 = S$.

Therefore f is constant on all of S.

Review Cramer's rule (from linear algebra).

Let $A \in \mathbb{R}^{n \times n}$ with $\det A \neq 0$. Then $Ax = b$ has a unique solution given by $x = (x_1, \ldots, x_n)$ where

$$x_i = \frac{\det A_i(b)}{\det A} \quad \text{for } i = 1, \ldots, n$$

and $A_i(b)$ is the matrix A with the ith column replaced by b.

More consequences of the chain rule ...

Let $F:\mathbb{R}^{n+1}\to\mathbb{R}$ differentiable and consider the solution set

$$\{x\in\mathbb{R}^{n+1} : F(x)=0\} = \{(x_1,\dots,x_n,y)\in\mathbb{R}^{n+1} : F(x_1,\dots,x_n,y)=0\}$$

Suppose there is a $g:S\to\mathbb{R}$ differentiable where $S\subseteq\mathbb{R}^n$ such that.

$$F(x_1,\dots,x_n,g(x_1,\dots,x_n))=0 \quad \text{for all } (x_1,\dots,x_n)\in S$$

or

$$F(x,g(x))=0 \quad \text{for all } x\in S.$$

$$\frac{\partial F(x,g(x))}{\partial x_i} = \partial_i F(x,g(x)) + \partial_{n+1}F(x,g(x))\,\partial_i g(x) = 0$$

solve for this

Therefore if $\partial_{n+1}F(x,g(x))\neq 0$ then

$$\partial_i g(x) = \frac{-\partial_i F(x,g(x))}{\partial_{n+1}F(x,g(x))}$$

can view condition on g.

or implicit way to find ∇g.

Idea want to differentiate a function $\varphi:\mathbb{R}^{n+1}\to\mathbb{R}$ subject to the constraint $F(x_1,\dots,x_n,y)=0$.

$$(x_1,\dots,x_n,y)=(x_1,\dots,x_n,g(x_1,\dots,x_n))$$

$$\frac{\partial\,\varphi(x,g(x))}{\partial x_i} = \partial_i\varphi(x,g(x)) + \partial_{n+1}\varphi(x,g(x))\,\partial_i g(x)$$

$$= \partial_i\varphi(x,g(x)) - \partial_{n+1}\varphi(x,g(x))\,\frac{\partial_i F(x,g(x))}{\partial_{n+1}F(x,g(x))}$$

Example: page 75

EXAMPLE 2. Let $w = x^2 + y^2 + z$, and suppose x, y, z are constrained to satisfy $x + y + z = 0$. If we take x and y as independent variables, then

$f(x,y,z) = x^2 + y^2 + z$ $\qquad$ $F(x,y,z) = x + y + z$

The constraint $F(x,y,z) = 0$ is satisfied

when $z = g(x,y) = -x-y$.

$$F(x,y,g(x,y)) = x+y-x-y = 0.$$

$$\left(\frac{\partial f}{\partial x}\right)_{y \text{ const}} = \frac{\partial f(x,y,g(x,y))}{\partial x} = \frac{\partial(x^2+y^2-x-y)}{\partial x} = \boxed{2x-1}$$

Note one could also solve for y in terms of x and z

$$y = -x-z$$

$$F(x,-x-z,z) = x-x-z+z = 0$$

$$\boxed{\left(\frac{\partial f}{\partial x}\right)_{z \text{ const}}} = \frac{\partial(x^2+(x+z)^2+z)}{\partial x} = 2x+2(x+z) = \boxed{4x+2z}$$

tricky point that must be confronted.

Generalized

$$F(x,y,u(x,y),v(x,y)) = 0$$
$$G(x,y,u(x,y),v(x,y)) = 0$$

for all $(x,y) \in S \subseteq \mathbb{R}^2$

assume $F: \mathbb{R}^4 \to \mathbb{R}$, $G: \mathbb{R}^4 \to \mathbb{R}$, $u: S \to \mathbb{R}$, $v: S \to \mathbb{R}$

are all differentiable.

$$\frac{\partial F(x,y,u(x,y),v(x,y))}{\partial x} =$$

$$\partial_1 F(x,y,u(x,y),v(x,y)) + \partial_3 F(x,y,u(x,y),v(x,y))\, \partial_1 u(x,y) + \partial_4 F(x,y,u(x,y),v(x,y))\, \partial_1 v(x,y) = 0$$

$$\frac{\partial G(x,y,u(x,y),v(x,y))}{\partial x} =$$

$$\partial_1 G(x,y,u(x,y),v(x,y)) + \partial_3 G(x,y,u(x,y),v(x,y))\, \partial_1 u(x,y) + \partial_4 G(x,y,u(x,y),v(x,y))\, \partial_1 v(x,y) = 0$$

Let

$$A = \begin{bmatrix} \partial_3 F(x,y,u(x,y),v(x,y)) & \partial_4 F(x,y,u(x,y),v(x,y)) \\ \partial_3 G(x,y,u(x,y),v(x,y)) & \partial_4 G(x,y,u(x,y),v(x,y)) \end{bmatrix} \quad b = \begin{bmatrix} -\partial_1 F(x,y,u(x,y),v(x,y)) \\ -\partial_1 G(x,y,u(x,y),v(x,y)) \end{bmatrix} \quad x = \begin{bmatrix} \partial_1 u(x,y) \\ \partial_1 v(x,y) \end{bmatrix}$$

Use Cramers rule to solve for $\partial_1 u$ and $\partial_1 v$.

$$\partial_1 u = \frac{\det A_1(b)}{\det A} = \frac{-\det \begin{bmatrix} \partial_1 F(x,y,u(x,y),v(x,y)) & \partial_4 F(x,y,u(x,y),v(x,y)) \\ \partial_1 G(x,y,u(x,y),v(x,y)) & \partial_4 G(x,y,u(x,y),v(x,y)) \end{bmatrix}}{\det \begin{bmatrix} \partial_3 F(x,y,u(x,y),v(x,y)) & \partial_4 F(x,y,u(x,y),v(x,y)) \\ \partial_3 G(x,y,u(x,y),v(x,y)) & \partial_4 G(x,y,u(x,y),v(x,y)) \end{bmatrix}}$$

$\partial_1 v$, $\partial_2 u$ and $\partial_2 v$ can be found the same way.

For next time read from the beginning of

2.7 Taylor's Theorem up to

2.68 Theorem (Taylor's Theorem in Several Variables). *Suppose* $f : \mathbb{R}^n \to \mathbb{R}$ *is of class* C^k *on an open convex set* S. *If* $\mathbf{a} \in S$ *and* $\mathbf{a} + \mathbf{h} \in S$, *then*