Example: Chain rule examples

$u = f(x,y)$ $\quad x = s^2 - t^2$ $\quad y = 2st$

need to assume f_x, f_y, f_{xy}, f_{yx} all exist and are continuous.

$$\frac{\partial u}{\partial t} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} = -2t f_x + 2s f_y$$

then Theorem 2.45 shows $f_{xy} = f_{yx}$.

$$\frac{\partial^2 u}{\partial s \partial t} = \frac{\partial}{\partial s}\left(-2t f_x + 2s f_y\right)$$

$$= -2t \frac{\partial}{\partial s} f_x + 2 f_y + 2s \frac{\partial}{\partial s} f_y$$

$$= -2t\left(f_{xx}\frac{\partial x}{\partial t} + f_{yx}\frac{\partial y}{\partial t}\right) + 2f_y + 2s\left(f_{yx}\frac{\partial x}{\partial s} + f_{yy}\frac{\partial y}{\partial s}\right)$$

$$= 4t^2 f_{xx} - \underbrace{4st f_{xy} + 4s^2 f_{xy}}_{\text{why } f_{xy} = f_{yx}?} + 4st f_{yy}$$

2.45 Theorem. *Let f be a function defined in an open set $S \subset \mathbb{R}^n$. Suppose $\mathbf{a} \in S$ and $i, j \in \{1, \ldots, n\}$. If the derivatives $\partial_i f$, $\partial_j f$, $\partial_i \partial_j f$, and $\partial_j \partial_i f$ exist in S, and if $\partial_i \partial_j f$ and $\partial_j \partial_i f$ are continuous at $\mathbf{a}$, then $\partial_i \partial_j f(\mathbf{a}) = \partial_j \partial_i f(\mathbf{a})$.*

Read proof at home... The definitions of the derivatives for f_{xy} and f_{yx} have h in different places, but in the limit that's okay because of the continuity.

$C^k(U)$

A function f is said to be of **class C^k** on an open set U if all of its partial derivatives of order $\le k$ — that is, all the derivatives $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_l} f$, for all choices of the indices i_j and all $l \le k$ — exist and are continuous on U. We also say that f is of class C^k on a nonopen set S if it is of class C^k on some open set that includes S. If the partial derivatives of f of *all* orders exist and are continuous on U, f is said to be of **class C^∞** on U.

Let $P_{a,k}(h) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j,$ and $R_{a,k}(h) = f(a+h) - P_{a,k}(h)$

2.55 Theorem (Taylor's Theorem with Integral Remainder, I). *Suppose that f is of class C^{k+1} ($k \geq 0$) on an interval $I \subset \mathbb{R}$, and $a \in I$. Then the remainder $R_{a,k}$ defined by* (2.53)–(2.54) *is given by*

$$R_{a,k}(h) = \frac{h^{k+1}}{k!} \int_0^1 (1-t)^k f^{(k+1)}(a+th)\,dt. \tag{2.56}$$

2.58 Theorem (Taylor's Theorem with Integral Remainder, II). *Suppose that f is of class C^k ($k \geq 1$) on an interval $I \subset \mathbb{R}$, and $a \in I$. Then the remainder $R_{a,k}$ defined by* (2.53)–(2.54) *is given by*

$$R_{a,k}(h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} \left[f^{(k)}(a+th) - f^{(k)}(a)\right] dt. \tag{2.59}$$

Proof By previous theorem. using k-1 we have

$$f(a+h) - \sum_{j=0}^{k-1} \frac{f^{(j)}(a)}{j!} h^j = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+th)\,dt$$

$$f(a+h) - \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+th)\,dt - \underbrace{\frac{f^{(k)}(a)}{k!} h^k}_{\text{put that inside the integral..}}$$

Since $\int_0^1 (1-t)^{k-1} dt = -\frac{1}{k}(1-t)^k \Big|_0^1 = \frac{1}{k}$

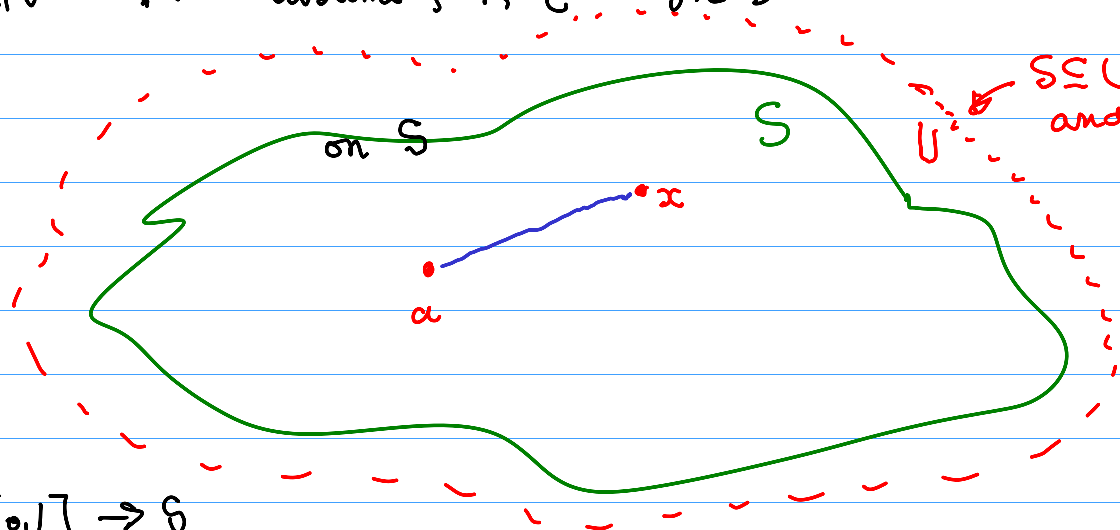
2.63 Theorem (Taylor's Theorem with Lagrange's Remainder). *Suppose f is $k+1$ times differentiable on an interval $I \subset \mathbb{R}$, and $a \in I$. For each $h \in \mathbb{R}$ such that $a+h \in I$ there is a point c between 0 and h such that*

$$R_{a,k}(h) = f^{(k+1)}(a+c)\frac{h^{k+1}}{(k+1)!}. \tag{2.64}$$

again from from Thm 2.55 ...

For the multi variable case we'll work along a line and apply the single variable result

$f: \mathbb{R}^n \to \mathbb{R}$ assume f is C^{k+1} on S



$S \subseteq U$ and U open

$g: [0,1] \to S$

$g(t) = f(a + t(x-a)) = f(a + th)$, $h = x - a \in \mathbb{R}^n$

Apply Theorem 2.55 to g.

Multi-index notation:

$x \in \mathbb{R}^n$ $\alpha \in (\mathbb{N} \cup \{0\})^n$

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_i \in \underbrace{\mathbb{N} \cup \{0\}}_{\text{non-negative integers}}$.

also define

$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$

$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$

use context to know what this means...

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

Example

$$(x, y, z)^{(2\ 0\ 7)} = x^2 z^7$$

$$\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f$$

$|\alpha|$ is order of this partial

$$\partial^{(2,0,7)} f(x,y,z) = \underbrace{\partial_1^2 \partial_3^7}_{\text{order } 9} f(x,y,z) = f_{xxzzzzzzz}$$

order 9 and $|(2,0,7)| = 9$

2.52 Theorem (The Multinomial Theorem). *For any* $\mathbf{x} = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$ *and any positive integer* k,

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathbf{x}^\alpha.$$

View as a binomial $((x_1 + \cdots + x_{n-1}) + x_n)^k$ and then use induction to prove the result.

Try to use $g(t)$ in Theorem 2.55

$$g(t) = f(a + ht)$$

$$g'(t) = \nabla f(a+ht) \cdot h = h \cdot \nabla f(a+ht)$$

$$= (h_1\partial_1 + h_2\partial_2 + \cdots + h_n\partial_n) f(a+ht) = (h \cdot \nabla) f(a+ht)$$

$$g^{(n)}(t) = (h \cdot \nabla)^n f(a+ht) = \underbrace{(h_1\partial_1 + h_2\partial_2 + \cdots + h_n\partial_n)^n}_{\text{now expand this by the multinomial theorem}} f(a+ht)$$

$$R_{a,k}(h) = \frac{h^{k+1}}{k!} \int_0^1 (1-t)^k f^{(k+1)}(a+th)\, dt.$$