The 2nd order Taylor's theorem was

$$f(a+h) = f(a) + h \cdot \nabla f(a) + \frac{1}{2} h \cdot Hh + R_2(h)$$

where $H \in \mathbb{R}^{n \times n}$ with entries $H_{ij} = \partial_i \partial_j f(a)$

Since differentiability of f implies $H = H^T$, then by the Spectral theorem $H = QDQ^T$ where $Q^T = Q^{-1}$ and

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix}$$ is the diagonal matrix of eigenvalues.

Suppose we're at a critical point. Then $\nabla f(a) = 0$.

To show $f(a)$ is a minimum, we need to show

$$\frac{1}{2} h \cdot Hh + R_2(h) \geq 0$$

for h small enough. To estimate from below

$$\frac{1}{2} h \cdot Hh = \frac{1}{2} h \cdot QDQ^T h = \frac{1}{2} \underbrace{Q^T h}_{c} \cdot DQ^T h = \frac{1}{2} c \cdot Dc$$

$$= \frac{1}{2} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \vdots \\ \lambda_n c_n \end{bmatrix} = \frac{1}{2} \sum_{i=1}^{n} \lambda_i c_i^2 \geq \frac{1}{2} \underbrace{(\min \lambda_i)}_{l} \sum c_i^2$$

$$= \frac{1}{2} l\, c \cdot c = \frac{1}{2} l\, Q^T h \cdot Q^T h = \frac{1}{2} l\, h \cdot \cancel{QQ^T} h = \frac{1}{2} l\, h \cdot h = \frac{1}{2} l |h|^2$$

In order to guarantee the bound from below is positive we require $l>0$.

Now, by the estimate on the remainder for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|h|<\delta \quad \text{implies} \quad |R_2(h)| \leq n^2|h|^2\varepsilon,$$

or that $\quad -|R_2(h)| \geq -n^2|h|^2\varepsilon.$

Thus, for $|h|<\delta$ we have

$$f(a+h) = f(a) + \tfrac{1}{2}h\cdot Hh + R_2(h) \geq f(a) + \tfrac{1}{2}l|h|^2 - |R_2(h)|$$
$$\geq f(a) + \tfrac{1}{2}l|h|^2 - n^2|h|^2\varepsilon = f(a) + \left(\tfrac{1}{2}l - n^2\varepsilon\right)|h|^2$$

Choosing $\varepsilon = \frac{l}{4n^2}$ then yields $\delta>0$ such that

$$f(a+h) \geq f(a) + \left(\tfrac{1}{2}l - n^2\tfrac{l}{4n^2}\right)|h|^2 = f(a) + \tfrac{1}{4}l|h|^2$$

for all $|h|<\delta$ where $\quad l = \min(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

We have proved the local minimum part of ...

2.81 Theorem. *Suppose f is of class C^2 at $\mathbf{a}$ and that $\nabla f(\mathbf{a}) = 0$, and let H be the Hessian matrix (2.79). For f to have a local minimum at $\mathbf{a}$, is it necessary for the eigenvalues of H all to be nonnegative and sufficient for them all to be strictly positive. For f to have a local maximum at $\mathbf{a}$, it is necessary for the eigenvalues of H all to be nonpositive and sufficient for them all to be strictly negative.*

The local maximum part is similar.

In the case $H \in \mathbb{R}^{2\times 2}$ then $H = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$

$$\det H = \alpha\gamma - \beta^2 = \lambda_1 \lambda_2$$

If $\lambda_1 > 0$ and $\lambda_2 > 0$ then $\det H > 0$.

If $\det H > 0$ then either $\lambda_1 > 0$ and $\lambda_2 > 0$

or $\lambda_1 < 0$ and $\lambda_2 < 0$.

$$\underbrace{\max(\lambda_1, \lambda_2)}|u|^2 \geq u \cdot Hu \geq \min(\lambda_1, \lambda_2)|u|^2$$

part not proved for max.

Take $u = e_1$ and suppose $\lambda_1 > 0$ and $\lambda_2 > 0$. Then

$$e_1 \cdot He_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha$$

$$\geq \min(\lambda_1, \lambda_2)|e_1|^2 = \min(\lambda_1, \lambda_2) > 0$$

which implies $\alpha > 0$. Similarly $\lambda_1 < 0$ and $\lambda_2 < 0$ and $\alpha < \max(\lambda_1, \lambda_2)$ implies $\alpha < 0$. Taking the contrapositive then implies ...

2.82 Theorem. *Suppose f is of class C^2 on an open set in $\mathbb{R}^2$ containing the point* $\mathbf{a}$, *and suppose $\nabla f(\mathbf{a}) = \mathbf{0}$. Let $\alpha = \partial_1^2 f(\mathbf{a})$, $\beta = \partial_1\partial_2 f(\mathbf{a})$, $\gamma = \partial_2^2 f(\mathbf{a})$. Then:*

a. *If $\underbrace{\alpha\gamma - \beta^2}_{\det} < 0$, f has a saddle point at* $\mathbf{a}$.

b. *If $\alpha\gamma - \beta^2 > 0$ and $\alpha > 0$, f has a local minimum at* $\mathbf{a}$.

c. *If $\alpha\gamma - \beta^2 > 0$ and $\alpha < 0$, f has a local maximum at* $\mathbf{a}$.

d. *If $\alpha\gamma - \beta^2 = 0$, no conclusion can be drawn.*

Recall: If $S \subseteq \mathbb{R}^n$ and S is compact and $f: S \to \mathbb{R}$ continuous, then f attains its min and max on S.

$\inf\{f(x) : x \in S\} = f(a)$ for some $a \in S$

$\sup\{f(x) : x \in S\} = f(b)$ for some $b \in S$.

Suppose $S = \overline{U}$ where $U \subseteq \mathbb{R}^n$ is open.

$$\partial S = \{x : G(x) = 0\}$$

for some $G \in C^1(\partial S)$ and $\nabla G(x) \neq 0$ for $x \in \partial S$.

called a smooth boundary...

also have piecewise smooth

$$\partial S = \bigcup_{i=1}^{N} \{x : G_i(x) = 0\}$$

save this complication for later.

Recall thing about level sets... I know that (lecture 10)

$\nabla G(x)$ is perpendicular to any tangent to ∂S at x.

$\nabla G(x)$ is normal to ∂S.

This is an application of the chain rule...

Suppose $a \in \partial S$ that is a minimum for f.

let $g : \mathbb{R} \to \partial S$ be such that $g(0) = a$ and $g \in C^1(\partial S)$ and $g'(0) \neq 0$.

Then $\varphi(t) = f(g(t))$ has a relative minimum at $t = 0$.

$\varphi : \mathbb{R} \to \mathbb{R}$.

This means $\varphi'(0) = 0$.

$$\varphi'(t) = \nabla f(g(t)) \cdot g'(t)$$

$$0 = \varphi'(0) = \nabla f(g(0)) \cdot g'(0) = \nabla f(a) \cdot g'(0)$$

So again we see $\nabla f(a)$ is perpendicular to any vector tangent to ∂S.

Since $\nabla G(a)$ and $\nabla f(a)$ are both normal to ∂S. then $\nabla G(a) = \lambda \nabla f(a)$ for some ratio λ.

Thus

$$\begin{cases} \partial_i G(a) = \lambda \partial_i f(a) & \text{for } i = 1, \dots, n \\ G(a) = 0 \end{cases}$$

also

This is a system of $n+1$ equations and λ and a represent $n+1$ unknowns...

In summary if $a \in S$ is a relative extrema for f the eithere

① $a \in U$ is an interior point and $\nabla f(a) = 0$

② $a \in \partial S$ then $G(a) = 0$ and there is λ such that $\nabla G(a) = \lambda \nabla f(a)$

For next time read 2.10...