## Implicit function theorem:

**3.3 Corollary.** *Let $F$ be a function of class $C^1$ on $\mathbb{R}^n$, and let $S = \{\mathbf{x} : F(\mathbf{x}) = 0\}$. For every $\mathbf{a} \in S$ such that $\nabla F(\mathbf{a}) \neq \mathbf{0}$ there is a neighborhood $N$ of $\mathbf{a}$ such that $S \cap N$ is the graph of a $C^1$ function.*

Different ways to describe a curve in $\mathbb{R}^2$.

i. as the graph of a function, $y = f(x)$ or $x = f(y)$, where $f$ is of class $C^1$;

easy

ii. as the locus[1] of an equation $F(x, y) = 0$, where $F$ is of class $C^1$;

prove this... $\nabla F \neq 0$ on $\{(x,y): F(x,y)=0\}$

iii. easy parametrically, as the range of a $C^1$ function $\boxed{\mathbf{f}}$: $(a, b) \to \mathbb{R}^2$.

$f'(t) \neq 0$ for $t \in (a,b)$

(i) $\Rightarrow$ (ii)

If $y = f(x)$ describes a curve, then

$\{(x,y): F(x,y)=0\}$ where $F(x,y) = y - f(x)$

describes the same curve.

(i) $\Rightarrow$ (iii)

If $y = f(x)$ describes a curve, then $\vec{f}(t) = (t, f(t))$

$\vec{f}(I) = \{(x, f(x)): x \in I\}$ for some interval $I$ also

describes the same curve.

(ii) $\Rightarrow$ (i)

Suppose $S = \{(x,y): F(x,y)=0\}$ describes a curve and that $\nabla F \neq 0$ on $S$. Then Corollary 3.3 implies

*For every $\mathbf{a} \in S$ such that $\nabla F(\mathbf{a}) \neq \mathbf{0}$ there is a neighborhood $N$ of $\mathbf{a}$ such that $S \cap N$ is the graph of a $C^1$ function.*

Suppose $\partial_x F \neq 0$ then by the theorem

$F(x,y)=0$ for $(x,y)\in S\cap N$ iff there is a function $f$ such that $\boxed{x=f(y)}$.

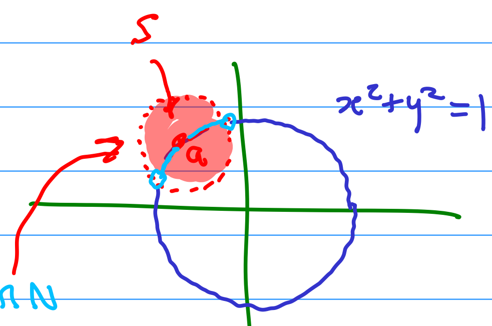
Suppose $\partial_y F \neq 0$ then by the theorem

$F(x,y)=0$ for $(x,y)\in S\cap N$ iff there is a function $f$ such that $\boxed{y=f(x)}$.

This here is the representation of the curve as a graph of a function as in (i) locally about each point on the curve.

Local representation of the curve by a graph does not mean there is a function whose graph gives the whole curve.

Example: $S=\{x^2+y^2=1\}$




on this neighborhood the curve is given by a graph..

$\partial_x(x^2+y^2-1) = 2x\big|_{(x,y)=a} \neq 0$ thus there is $x=f(y)$ on $S\cap N$

$x=-\sqrt{1-y^2}$

$\partial_y(x^2+y^2-1) = 2y\big|_{(x,y)=a} \neq 0$ thus there is $y=f(x)$ on $S\cap N$

$y=\sqrt{1-x^2}$

Now consider a curve represented as in (iii) and try to write it locally as (i).

Suppose the curve is given.

iii. ~~parametrically~~ easy / prove this... parametrically, as the range of a $C^1$ function $\boxed{\mathbf{f}}: (a, b) \to \mathbb{R}^2$. $\nabla F \neq 0$ on $\{(x,y): F$

and also $\underline{\vec{f}'(t) \neq 0 \quad \text{for } t \in (a,b)}$.

Claim that at any point $t_0$ the curve is locally a graph.

Let $\vec{f}(t) \doteq (\varphi(t), \psi(t))$

$\vec{f}(t_0) \neq 0$ means either $\varphi'(t_0) \neq 0$ or $\psi'(t_0) \neq 0$ or both.

Suppose $\varphi'(t_0) \neq 0$. Then define $F(x,t) = x - \varphi(t)$ and consider $S = \{(x,t): F(x,t) = 0\}$.

in effect this is going to invert $\varphi$.

Let $x_0 = \varphi(t_0)$

$$\partial_t F(x_0, t_0) = -\varphi'(t_0) \neq 0$$

So by Corollary 3.3 or the original implicit function theorem there is a function $w$ such that $t = w(x)$ in a neighborhood of $(x_0, t_0)$. This means on that neighborhood $N$ of $x_0$

$$0 = F(x, w(x)) = x - \varphi(w(x))$$

Consider the curve on a small enough neighborhood $I$ of $t_0$. such that $\varphi(t) \in N$ for all $t \in I$.

Thus $\vec{f}(I) = \{(\varphi(t), \psi(t)) : t \in I\}$.

For $t \in I$ we have $(\varphi(t), \psi(t)) = (x, \psi(w(x)))$ for some $x \in N$.

That means this part of the curve is the graph of the function $f(x) = \psi(w(x))$ locally.