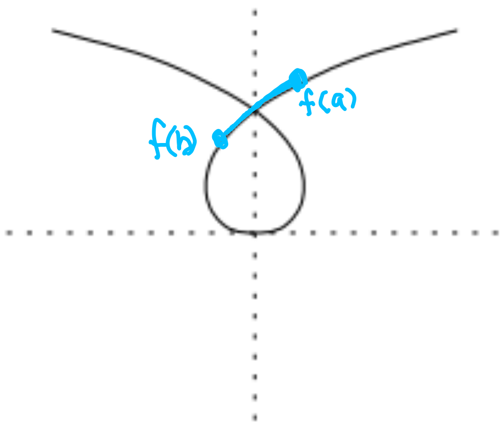

$f : (a,b) \to \mathbb{R}$

for this part of the curve it can be represented as a graph...

Surfaces in $\mathbb{R}^3$**.** The standard ways of representing surfaces in 3-space are analogous to the standard ways of representing curves in the plane:

easy

i. as the graph of a function, $z = f(x,y)$ (or $y = f(x,z)$ or $x = f(y,z)$), where f is of class C^1; $f : \mathbb{R}^2 \to \mathbb{R}$

difficult

ii. as the locus of an equation $F(x,y,z) = 0$, where F is of class C^1;

Curve $S = \{(x,y,z) : F(x,y,z) = 0\}$ need $\nabla F(x,y,z) \neq 0$ for $(x,y,z) \in S$.

iii. parametrically, as the range of a C^1 function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$.

Let $U \subseteq \mathbb{R}^2$ with U open   the surface is the range $f(U)$

for (iii) to be seen as a graph need

the vectors $\dfrac{\partial \mathbf{f}}{\partial u}(u,v)$ and $\dfrac{\partial \mathbf{f}}{\partial v}(u,v)$ are linearly independent

at each $(u,v) \in U$.

In the case of curves (a,b) is open

$\mathbf{f} : (a,b) \to \mathbb{R}^2$.

3.15 Theorem.

a. *Let F be a real-valued function of class C^1 on an open set in $\mathbb{R}^3$, and let $S = \{(x,y,z) : F(x,y,z) = 0\}$. If $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq \mathbf{0}$, there is a neighborhood N of $\mathbf{a}$ in $\mathbb{R}^3$ such that $S \cap N$ is the graph of a C^1 function f (either $z = f(x,y)$, $y = f(x,z)$, or $x = f(y,z)$).*

b. *Let $\mathbf{f}$ be a C^1 mapping from an open set in $\mathbb{R}^2$ into $\mathbb{R}^3$. If $[\partial_u \mathbf{f} \times \partial_v \mathbf{f}](u_0, v_0) \neq \mathbf{0}$, there is a neighborhood N of (u_0, v_0) in $\mathbb{R}^2$ such that the set $\{\mathbf{f}(u,v) : (u,v) \in N\}$ is the graph of a C^1 function.*

$$\text{the vectors } \frac{\partial \mathbf{f}}{\partial u}(u,v) \text{ and } \frac{\partial \mathbf{f}}{\partial v}(u,v) \text{ are linearly independent at each } (u,v) \in U.$$

Can write this independence condition as

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix} \qquad \frac{\partial f}{\partial v} = \begin{bmatrix} \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial v} \end{bmatrix}.$$

These vectors are independent if:

$$\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix} \times \begin{bmatrix} \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial v} \end{bmatrix} \neq 0$$

also

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{bmatrix}.$$

since $f \in C^1(\mathcal{U})$ then this is the derivative of f.

rank $Df = 2$ the columns are independent

Proof. Part (a) is a special case of Corollary 3.3. As for (b), let $\mathbf{f} = (\varphi, \psi, \chi)$. The components of the cross product $\partial_u \mathbf{f} \times \partial_v \mathbf{f}$ are just the Jacobians $\partial(\varphi, \psi)/\partial(u,v)$, $\partial(\varphi, \chi)/\partial(u,v)$, and $\partial(\psi, \chi)/\partial(u,v)$. Under the hypothesis of (b), at least one of them — let us say $\partial(\varphi, \psi)/\partial(u,v)$ — is nonzero at (u_0, v_0). The implicit function theorem then guarantees that the pair of equations $x = \varphi(u,v)$, $y = \psi(u,v)$ can be solved to yield u and v as C^1 functions of x and y near $u = u_0$, $v = v_0$, $x = \varphi(u_0, v_0)$, $y = \psi(u_0, v_0)$. Substituting these functions for u and v in the equation $z = \chi(u,v)$ then yields z as a C^1 function of x and y whose graph is the range of $\mathbf{f}$. $\square$

i. as a graph, $y = f(x)$ and $z = g(x)$ (or similar expressions with the coordinates permuted), where f and g are C^1 functions;

easy

ii. as the locus of two equations $F(x, y, z) = G(x, y, z) = 0$, where F and G are C^1 functions;

iii. parametrically, as the range of a C^1 function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$.

(i) → (ii)

$$F(x,y,z) = y - f(x) \qquad G(x,y,z) = z - g(x)$$

Then $F(x,y,z) = 0$ means $y = f(x)$

$G(x,y,z) = 0$ means $z = g(x)$

Therefore $S = \{(x,y,z) : F(x,y,z) = 0\} \cap \{(x,y,z) : G(x,y,z) = 0\}$

are the solutions to $F(x,y,z) = G(x,y,z) = 0$ which means $y = f(x)$ and $z = g(x)$ for every $(x,y,z) \in S$.

(i) → (iii)

$$\vec{f}(x) = (x, f(x), g(x))$$

then the image is the curve defined by

$y = f(x)$ and $z = g(x)$

Need more general implicit function theorem ...

3.9 Theorem (The Implicit Function Theorem for a System of Equations)**.** *Let* $\mathbf{F}(\mathbf{x}, \mathbf{y})$ *be an* $\mathbb{R}^k$*-valued function of class* C^1 *on some neighborhood of a point* $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+k}$ *and let* $B_{ij} = (\partial F_i / \partial y_j)(\mathbf{a}, \mathbf{b})$*. Suppose that* $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ *and* $\det B \neq 0$*. Then there exist positive numbers* r_0, r_1 *such that the following conclusions are valid.*

a. *For each* $\mathbf{x}$ *in the ball* $|\mathbf{x} - \mathbf{a}| < r_0$ *there is a unique* $\mathbf{y}$ *such that* $|\mathbf{y} - \mathbf{b}| < r_1$ *and* $\mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$*. We denote this* $\mathbf{y}$ *by* $\mathbf{f}(\mathbf{x})$*; in particular,* $\mathbf{f}(\mathbf{a}) = \mathbf{b}$*.*

b. *The function* $\mathbf{f}$ *thus defined for* $|\mathbf{x} - \mathbf{a}| < r_0$ *is of class* C^1*, and its partial derivatives* $\partial_{x_j} \mathbf{f}$ *can be computed by differentiating the equations* $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$ *with respect to* x_j *and solving the resulting linear system of equations for* $\partial_{x_j} f_1, \ldots, \partial_{x_j} f_k$*.*

Proof in appendix: B2

B.2 The Implicit Function Theorem

B.2 Theorem. *Let* $\mathbf{F}(\mathbf{x}, \mathbf{y})$ *be an* $\mathbb{R}^k$*-valued function of class* C^1 *on some ne-*
borhood of a point $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+k}$ *and let* $B_{ij} = (\partial F_i / \partial y_j)(\mathbf{a}, \mathbf{b})$*. Suppose*

please read this proof after the exam over the weekend ...