moved from the last slot to the jth slot. The determinant of this matrix is therefore $(-1)^{k-j}\det M^{kj}$ — one factor of -1 because of the minus signs on the column of B_{ik}'s, and $k-j-1$ more factors of -1 from interchanging that column with the succeeding $k-j-1$ columns to move it back to its rightful place on the right end. In short, the application of Cramer's rule to the system (B.5) yields

$$\frac{\partial g_j}{\partial y_k}(\mathbf{a}, b_k) = (-1)^{k-j}\frac{\det M^{jk}}{\det M^{kk}}.$$

Now we are done. Substitute this result back into (B.4), noting that $(-1)^{-j} = (-1)^j$, and recall (B.3):

self correcting typo...

$$\frac{\partial G}{\partial y_k}(\mathbf{a}, b_k) = \sum_{j=1}^{k-1}(-1)^{j+k}B_{kj}\frac{\det M^{kj}}{\det M^{kk}} + B_{kk}$$
$$= \frac{\sum_{j=1}^{k}(-1)^{j+k}B_{kj}\det M^{kj}}{\det M^{kk}} = \frac{\det B}{\det M^{kk}}.$$

3.9 Theorem (The Implicit Function Theorem for a System of Equations). *Let $\mathbf{F}(\mathbf{x}, \mathbf{y})$ be an $\mathbb{R}^k$-valued function of class C^1 on some neighborhood of a point $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+k}$ and let $B_{ij} = (\partial F_i/\partial y_j)(\mathbf{a}, \mathbf{b})$. Suppose that $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ and $\boxed{\det B \neq 0}$. Then there exist positive numbers r_0, r_1 such that the following conclusions are valid.*

a. *For each $\mathbf{x}$ in the ball $|\mathbf{x} - \mathbf{a}| < r_0$ there is a unique $\mathbf{y}$ such that $|\mathbf{y} - \mathbf{b}| < r_1$ and $\mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$. We denote this $\mathbf{y}$ by $\mathbf{f}(\mathbf{x})$; in particular, $\mathbf{f}(\mathbf{a}) = \mathbf{b}$.*

b. *The function $\mathbf{f}$ thus defined for $|\mathbf{x}-\mathbf{a}| < r_0$ is of class C^1, and its partial derivatives $\partial_{x_j}\mathbf{f}$ can be computed by differentiating the equations $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$ with respect to x_j and solving the resulting linear system of equations for $\partial_{x_j} f_1, \ldots, \partial_{x_j} f_k$.*

Note $(a,b) \in \mathbb{R}^{n+k}$ means $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^k$

$\mathcal{U} \subseteq \mathbb{R}^{n+k}$, $F : \mathcal{U} \to \mathbb{R}^k$

$$B = [B_{ij}] = \left[\frac{\partial F_i}{\partial y_j}(a,b)\right] = \begin{bmatrix} \partial F_1/\partial y_1 & \cdots & \partial F_1/\partial y_k \\ \vdots & & \vdots \\ \partial F_k/\partial y_1 & \cdots & \partial F_k/\partial y_k \end{bmatrix}$$

$F(x,y) = 0$

$F(a,b) = 0$ ...

use this to define y as a function of x...

Proof: By induction on k.

k=1 is Theorem 3.1 (base case)

Assume the theorem is true for k-1 and try to prove it for k.

$$B = \begin{bmatrix} \partial F_1/\partial y_1 & \cdots & \partial F_1/\partial y_k \\ \vdots & & \vdots \\ \partial F_k/\partial y_1 & \cdots & \partial F_k/\partial y_k \end{bmatrix}$$ and $\det B = 0$

where are they evaluated? Let (a,b) be a point such that $F(a,b)=0$.

$$\det B = \sum_{j=1}^{k} (-1)^{k+j} B_{kj} \det M^{kj}$$

where M^{kj} is the cofactor matrix of B given by crossing out the k^{th} row and j^{th} column..

since det B is non-zero at least one term in the sum is non-zero...

Suppose $B_{kk} \det M^{kk} \neq 0$ (for convenience of notation).

$$M^{kk} = \begin{bmatrix} \partial F_1/\partial y_1 & \cdots & \partial F_1/\partial y_{k-1} & \partial F_1/\partial y_k \\ \vdots & & \vdots & \\ \partial F_{k-1}/\partial y_1 & \cdots & \partial F_{k-1}/\partial y_{k-1} & \\ \partial F_k/\partial y_1 & \cdots & & \partial F_k/\partial y_k \end{bmatrix}\Bigg|_{(x,y)=(a,b)}$$

where are they evaluated? Let (a,b) be a point such that $F(a,b)=0$.

$\det M^{kk} = 0$

$$F_1(x, y_1, \ldots, y_{k-1}, y_k) = F_2(x, y_1, \ldots, y_{k-1}, y_k) = \ldots = F_{k-1}(x, y_1, \ldots, y_{k-1}, y_k) = 0$$

by induction hypothesis:

$$\begin{bmatrix} F_1 \\ \vdots \\ F_{k-1} \end{bmatrix} : \mathbb{R}^{n+1+(k-1)} \to \mathbb{R}^{k-1}$$

there exist a function $g : \mathbb{R}^{n+1} \to \mathbb{R}^{k-1}$ such that

$$F_1(x, g(x, y_k), y_k) = F_2(x, g(x, y_k), y_k)$$
$$= F_{k-1}(x, g(x, y_k), y_k) = 0$$

To solve for all the y's in terms of x, we need to solve for y_k in terms of x. Theorem 3.1 again applies.

We want to use $F_k(x, g(x, y_k), y_k) = 0$ to solve for y_k.

Hypothesis needed $\partial_y F(\mathbf{a}, b) \neq 0.$

Need to show: (use the hypothesis $\boxed{\det B \neq 0}$)

$$\frac{\partial F_k(x, g(x, y_k), \bar{y}_k)}{\partial y_k} \neq 0$$

By the Chain rule

$$\frac{\partial F_k(x, g(x, y_k), \bar{y}_k)}{\partial y_k} = \sum_{j=1}^{k-1} \underbrace{\boxed{\frac{\partial F_k}{\partial y_j}}(x, g(x, y_k), y_k)}_{\text{means } \left.\frac{\partial F_k(x,y)}{\partial y_j}\right|_{(x,y) = (x, g(x, y_k), y_k)}} \frac{\partial g_j(x, y_k)}{\partial y_k} + \boxed{\frac{\partial F_k}{\partial y_k}}(x, g(x, y_k), y_k)$$

$$\left.\frac{\partial F_k(x, g(x,y_k), \bar{y_k})}{\partial y_k}\right|_{(x,y_k)=(a,b_k)} = \sum_{j=1}^{k-1} B_{kj} \frac{\partial g_j}{\partial y_k}(a, b_k) + B_{kk}$$

need to write this in terms of the B ...

(implicit differentiation)

Recall

$$F_1(x, g(x,y_k), y_k) = F_2(x, g(x,y_k), y_k)$$

$$= F_{k-1}(x, g(x,y_k), y_k) = 0$$

$$\frac{\partial F_i(x, g(x,y_k), y_k)}{\partial y_k} = \sum_{j=1}^{k-1} \boxed{\frac{\partial F_i}{\partial y_j}}(x, g(x,y_k), y_k) \frac{\partial g_j(x,y_k)}{\partial y_k} + \boxed{\frac{\partial F_i}{\partial y_k}}(x, g(x,y_k), y_k)$$

$$\left.\frac{\partial F_i(x, g(x,y_k), y_k)}{\partial y_k}\right|_{(x,y_k)=(a,b_k)} = \sum_{j=1}^{k-1} B_{ij} \frac{\partial g_j}{\partial y_k}(a, b_k) + B_{ik} = 0 \qquad \text{for } i = 1, \dots, k-1$$

solve for $\frac{\partial g_j}{\partial y_k}$

System of linear equation

$$\underbrace{\begin{bmatrix} B_{11} & \cdots & B_{1,k-1} \\ \vdots & \ddots & \vdots \\ B_{k-1,1} & \cdots & B_{k-1,k-1} \end{bmatrix}}_{M^{kk}} \begin{bmatrix} \partial g_1/\partial y_k \\ \vdots \\ \frac{\partial g_{k-1}}{\partial y_k} \end{bmatrix} = - \begin{bmatrix} B_{1k} \\ \vdots \\ B_{k-1,k} \end{bmatrix}$$

$\det M^{kk} \neq 0$

Cramer's rule,

$$\partial g_j / \partial y_k = \frac{\det M^{kk}_j \left(-\begin{bmatrix} B_{1k} \\ \vdots \\ B_{k-1,k} \end{bmatrix} \right)}{\det M^{kk}}$$

replace the jth column of M^{kk} with $-\begin{bmatrix} B_{1k} \\ \vdots \\ B_{k-1,k} \end{bmatrix}$

Substitute into

$$\left. \frac{\partial F_k(x, g(x, y_k), \bar{y}_k)}{\partial y_k} \right|_{(x, y_k) = (a, b_k)} = \sum_{j=1}^{k-1} B_{kj} \frac{\partial g_j}{\partial y_k}(a, b_k) + B_{kk}$$

d to write.

Next time finish this and then start integration