(ordered)

$\boxed{\text{a partition of } [a,b]}$ is a set $\{x_0, x_1, \ldots, x_J\}$

such that $a = x_0 < x_1 < \cdots < x_J = b$.

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$$U(P,f) = \sum_{j=1}^{J} M_j \Delta x_j = S_P(f)$$

$M_j := \sup\{f(x) : x \in [x_{j-1}, x_j]\}$

definition makes sense when f is bounded.

$\Delta x_j := x_j - x_{j-1}$

$$L(P,f) = \sum_{j=1}^{J} m_j \Delta x_j = s_P(f)$$

$m_j := \inf\{f(x) : x \in [x_{j-1}, x_j]\}$

$$\underline{I}_a^b f = \underline{\int_a^b} f = \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}$$

$$\overline{I}_a^b f = \overline{\int_a^b} f = \inf\{U(P,f) : P \text{ is a partition of } [a,b]\}$$

A function $f : [a,b] \to \mathbb{R}$ is said to be Reimann integrable if $\underline{I}_a^b f = \overline{I}_a^b f$ in which case we define

lower Reimann integral

$$\int_a^b f = \underline{I}_a^b f = \overline{I}_a^b f$$

upper Reimann integral.

Reimann integral is the common value.

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4.5 Lemma. *If f is a bounded function on $[a,b]$, the following conditions are equivalent:*

a. *f is integrable on $[a,b]$.*

b. *For every $\epsilon > 0$ there is a partition P of $[a,b]$ such that $S_P f - s_P f < \epsilon$.*

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Theorem 6.2 A bounded function f is in $\mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

4.6 Theorem.

assume integrable on smaller intervals

a. *Suppose $a < b < c$. If f is integrable on $[a,b]$ and on $[b,c]$, then f is integrable on $[a,c]$, and*

$$\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.} \tag{4.7}$$

b. *If f and g are integrable on $[a,b]$, then so is $f+g$, and*

$$\boxed{\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.} \tag{4.8}$$

assumed integrable on bigger one!

Proposition 6.5 Let f be in $\mathcal{R}[a,b]$ and let $a < c < b$. Then f is in $\mathcal{R}[a,c]$, f is in $\mathcal{R}[c,b]$, and $\boxed{\int_a^b f = \int_a^c f + \int_c^b f.}$

Proposition 6.3 Let f and g be in $\mathcal{R}[a,b]$ and let c be in $\mathbb{R}$. Then $f \pm g$ and cf are in $\mathcal{R}[a,b]$, and

$$\boxed{\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g}$$

and

$$\boxed{\int_a^b cf = c\int_a^b f.}$$

Same.

4.9 Theorem. *Suppose f is integrable on $[a,b]$.*

a. *If $c \in \mathbb{R}$, then cf is integrable on $[a,b]$, and* $\boxed{\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.}$

* b. *If $[c,d] \subset [a,b]$, then f is integrable on $[c,d]$.*

c. *If g is integrable on $[a,b]$ and $f(x) \le g(x)$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.*

* d. *$|f|$ is integrable on $[a,b]$, and $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.*

a<b

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4.10 Theorem. *If f is bounded and monotone on $[a,b]$, then f is integrable on $[a,b]$.*

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Theorem 6.9 If f is monotone on $[a, b]$, then f is in $\mathcal{R}[a, b]$.

4.11 Theorem. *If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.*

Theorem 6.7 If f is continuous on $[a, b]$, then f is in $\mathcal{R}[a, b]$.

4.12 Theorem. *If f is bounded on $[a,b]$ and continuous at all except finitely many points in $[a,b]$, then f is integrable on $[a,b]$.*

Theorem 6.5 Suppose that f is in $\mathcal{R}[a, b]$ and that $g = f$ except at a finite number of points in $[a, b]$. Then g is in $\mathcal{R}[a, b]$ and $\int_a^b g = \int_a^b f$.

implies the other is integrable

both integrable

4.14 Proposition. *Suppose f and g are integrable on $[a,b]$ and $f(x) = g(x)$ for all except finitely many points $x \in [a,b]$. Then $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.*

please. Certain infinite sets, such as convergent sequences, also have this property (Exercise 6). We make it into a formal definition: A set $Z \subset \mathbb{R}$ is said to have **zero content** if for any $\epsilon > 0$ there is a finite collection of intervals $I_1, \ldots, I_L$ such that (i) $Z \subset \bigcup_1^L I_l$, and (ii) the sum of the lengths of the I_l's is less than ϵ. The proof of Theorem 4.12 now yields the following result:

4.13 Theorem. *If f is bounded on $[a,b]$ and the set of points in $[a,b]$ at which f is discontinuous has zero content, then f is integrable on $[a,b]$.*

Need the idea of zero content in 2D and higher because, for example a line has no area but has infinite # of points

The starting point is the analogue of Theorem 4.13. The notion of "zero content" transfers readily to sets in the plane; namely, a set $Z \subset \mathbb{R}^2$ is said to have **zero content** if for any $\epsilon > 0$ there is a finite collection of rectangles $R_1, \ldots, R_M$ such that (i) $Z \subset \bigcup_1^M R_m$, and (ii) the sum of the areas of the R_m's is less than ϵ. We then have:

4.18 Theorem. *Suppose f is a bounded function on the rectangle R. If the set of points in R at which f is discontinuous has zero content, then f is integrable on R.*

4.15 Theorem (The Fundamental Theorem of Calculus).

does $F(x)$ exist? small set $[a,x)$

large set

a. *Let f be an integrable function on $[a,b]$. For $x \in [a,b]$, let $F(x) = \int_a^x f(t)\,dt$ (which is well defined by Theorem 4.9b). Then F is continuous on $[a,b]$; moreover, $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous.*

b. *Let F be a continuous function on $[a,b]$ that is differentiable except perhaps at finitely many points in $[a,b]$, and let f be a function on $[a,b]$ that agrees with F' at all points where the latter is defined. If f is integrable on $[a,b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.*

Part(a) Let $x, y \in [a,b]$. Then

$$F(x) - F(y) = \int_a^x f(t)dt - \int_a^y f(t)\,dt \overset{\text{theorem 4.6}}{=} \int_y^x f(t)\,dt$$

Assume $x > y$ then theorem 4.9d gives on the interval $[y, x]$

$$\left|F(x) - F(y)\right| = \left|\int_y^x f(t)\,dt\right| \leq \int_y^x |f(t)|\,dt \leq \int_y^x C\,dt = (x-y)\,C$$

Since $f \in \mathcal{R}[a,b]$ then f is bounded.

set of all Reimann integrable functions on $[a,b]$

Let C be the bound so that

$$|f(t)| \leq C \quad \text{for all } t \in [a,b].$$

If $x < y$ the same argument shows

$$\left|F(x) - F(y)\right| \leq \int_x^y |f(t)|\,dt \leq \int_x^y C\,dt = (y-x)\,C$$

together gives.

Therefore

$$\left|F(x) - F(y)\right| \leq |x-y|\,C$$

This means F is continuous...

$$\frac{F(y) - F(x)}{y-x} = \frac{1}{y-x}\int_x^y f(t)\,dt$$

$$f(x) = f(x)\,\frac{1}{y-x}\int_x^y dt = \frac{1}{y-x}\int_x^y f(x)\,dt$$

const wrt. t

$$\frac{F(y) - F(x)}{y-x} - f(x) = \frac{1}{y-x}\int_x^y \big(f(t) - f(x)\big)\,dt$$

$$\left| \frac{F(y)-F(x)}{y-x} - f(x) \right| \leq \frac{1}{|y-x|} \left| \int_x^y |f(t)-f(x)|\, dt \right|$$

Suppose f is continuous at x. Fix x.

$\forall \varepsilon > 0 \ \exists \delta > 0$ st $|t-x| < \delta$ implies $|f(t)-f(x)| < \varepsilon$