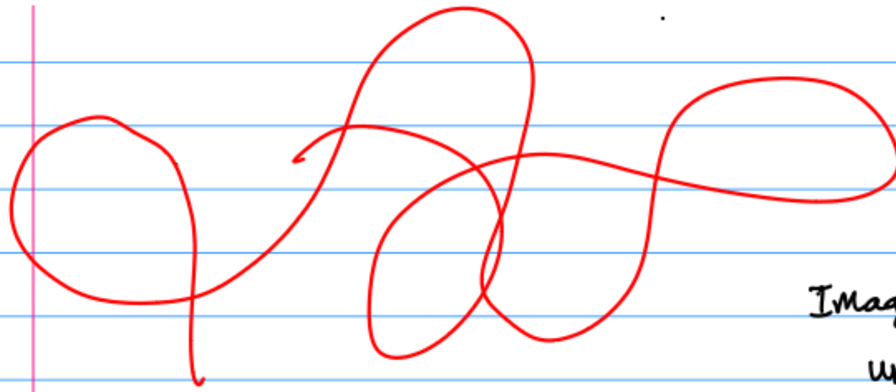
c. *If* $\mathbf{f} : (a_0, b_0) \to \mathbb{R}^2$ *is of* $\boxed{\text{class } C^1,}$ *then* $\mathbf{f}([a,b])$ *has zero content whenever* $a_0 < a < b < b_0$.



has zero content in $\mathbb{R}^2$

Image of a closed interval under a C^1 function.

The starting point is the analogue of Theorem 4.13. The notion of "zero content" transfers readily to sets in the plane; namely, a set $\underline{Z} \subset \mathbb{R}^2$ is said to have **zero content** if for any $\epsilon > 0$ there is a finite collection of rectangles $R_1, \ldots, R_M$ such that (i) $Z \subset \bigcup_1^M R_m$, and (ii) the sum of the areas of the R_m's is less than ϵ. We then have:

Note that $[a,b]$ is compact so $f:[a,b] \to \mathbb{R}^2$ is uniformly continuous...

Also $f([a,b])$ is compact so f is bounded on $[a,b]$.

Write $f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$ and $f'(t) = \begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix}$

Since $f \in C^1([a,b])$ then f' is cont. there is a bound c such that

$$\underline{|f_1'(t)| \leq c} \quad \text{and} \quad \underline{|f_2'(t)| \leq c} \quad \text{for all } t \in [a,b].$$

let's divide $[a,b]$ into k equal subintervals

$$t_0 = a \qquad t_j = t_0 + \delta j \quad \text{where} \quad \delta = \frac{b-a}{k}$$

The subintervals are $[t_{j-1}, t_j]$ for $j = 1, \ldots, k$.

By the mean value theorem...

$$|f_1(t) - f_1(t_j)| = |f_1'(c)(t - t_j)|$$ for some c between t and t_j

$$|f_2(t) - f_2(t_j)| = |f_2'(c)(t - t_j)|$$ for some c between t and t_j

Therefore if $t \in [t_{j-1}, t_j]$ then

$$|f_1(t) - f_1(t_j)| \le C|t - t_j| \quad \text{and} \quad |f_2(t) - f_2(t_j)| \le C|t - t_j|$$

or

$$|f_1(t) - f_1(t_j)| \le C\delta \qquad |f_2(t) - f_2(t_j)| \le C\delta$$

mean value theorem gives a δ here...

Define the rectangle

$$R_j = \{(y_1, y_2) : |y_1 - f_1(t_j)| \le C\delta \text{ and } |y_2 - f_2(t_j)| \le C\delta\}$$

Thus $f(t) = (f_1(t), f_2(t)) \in R_j$ for $t \in [t_{j-1}, t_j]$.

$$f([a,b]) = \bigcup_{j=1}^{K} f([t_{j-1}, t_j]) \subseteq \bigcup_{j=1}^{k} R_j$$

Need to show the sum of the areas of the R_j is as small as I want to claim $f([a,b])$ has zero content.

$$\sum_{j=1}^{k} |R_j| = \sum_{j=1}^{k} \underbrace{2C\delta}_{\text{width}} \cdot \underbrace{2C\delta}_{\text{height}} = k\,4C^2\delta^2$$

Recall

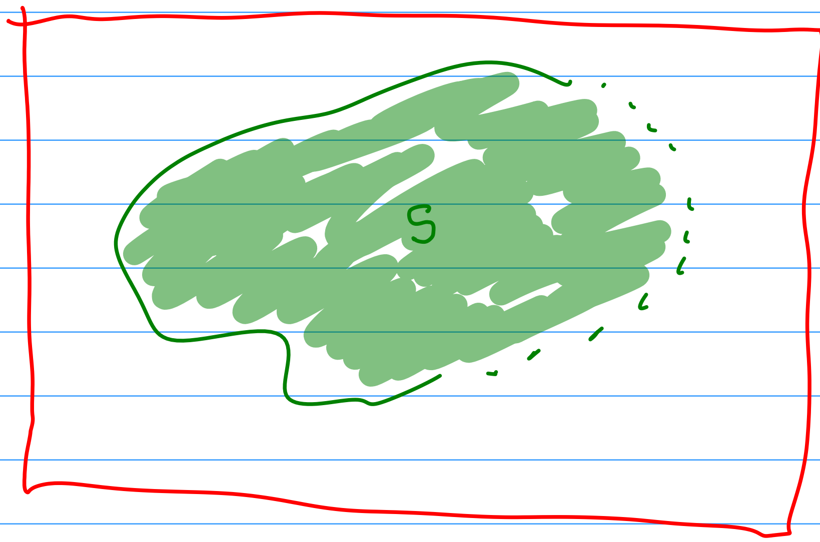
$$\delta = \frac{b-a}{k}$$

Then

$$\sum_{j=1}^{k} |R_j| = k4C^2\delta^2 = k4C^2\left(\frac{b-a}{k}\right)^2 = \frac{4C^2(b-a)^2}{k}$$

choosing k large shows this area can be made arbitrarily small... Thus $f([a,b])$ has zero content.

$R = [a,b] \times [c,d]$



To integrate over S we define

$$\iint_S f = \iint_R f \chi_S$$

Suppose f were discontinuous on the set Z_1.
and χ_S were discontinuous on the set Z_2
(note Z_2 is the boundary of S)

The function $g(x) = f(x)\chi_S(x)$ is discontinuous where?

or here

discontinuous here

The set of discontinuities of g is contained in $Z_1 \cup Z_2$.

4.21 Theorem. *Let S be a measurable subset of $\mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded and the set of points in S at which f is discontinuous has zero content. Then f is integrable on S.*

Proof. The only points where $f\chi_S$ can be discontinuous are those points in the closure of S where either f or χ_S is discontinuous. By Lemma 4.20 and Proposition 4.19b, the set of such points has zero content. By Theorem 4.18, $\underbrace{f\chi_S}_{g}$ is integrable on any rectangle R containing S, and hence f is integrable on S. $\square$

4.18 Theorem. *Suppose g is a bounded function on the rectangle R. If the set of points in R at which g is discontinuous has zero content, then f is integrable on R.*

Proof. The proof is essentially identical to that of Theorem 4.13. That is, one first shows that g is integrable if g is continuous on all of R by the argument that proves Theorem 4.11, then encompasses the general case by the argument that proves Theorem 4.12. Details are left to the reader. $\square$

$R = [a,b] \times [c,d]$.

Since g is bounded there is a bound C such that

$$|g(x)| \leq C \quad \text{for every } x \in R.$$

Let $\varepsilon > 0$. Claim there is a partition

$$P = \{x_0, x_1, \ldots, x_J ; y_0, y_1, \ldots, y_K\}$$

Such that $\quad |s_P g - S_P g| < \varepsilon$.

Choose $\varepsilon_1 =$ ________. Then by the definition of zero content there are open rectangles R_m such that

$$Z \subseteq \bigcup_{m=1}^{M} R_m \qquad \text{where } Z = \{x \in R : g \text{ is not cont. at } x\}.$$

and $\displaystyle\sum_{m=1}^{M} |R_m| < \varepsilon_1$.

closed

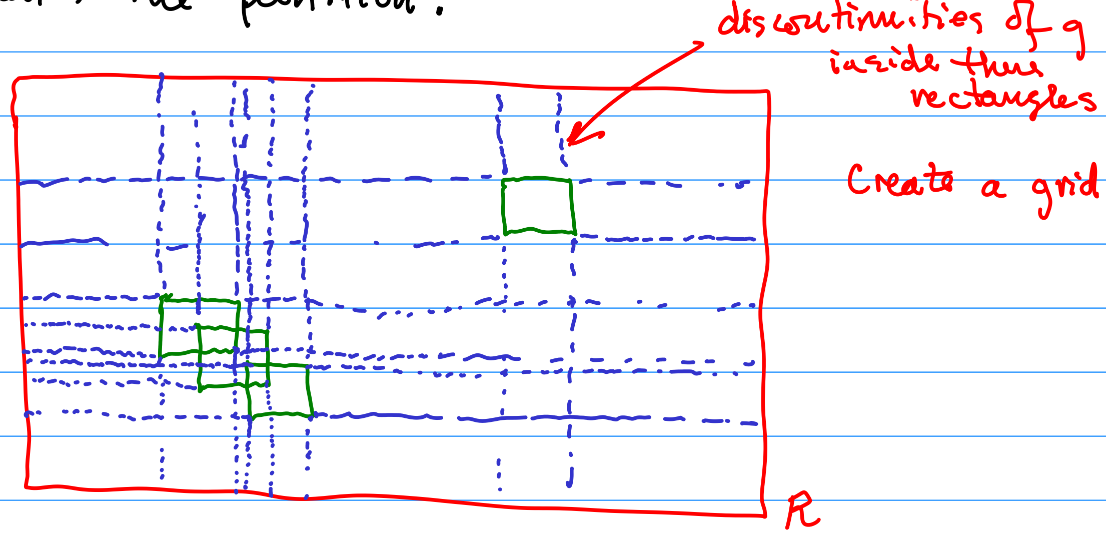
complement of an open set

Note that $U = \bigcup_{m=1}^{M} R_m$ is open so $V = R \setminus U$ is closed.

Since the discontinuities of g are all in U then g is continuous on the set V. Since V is closed and bounded then V is compact. Therefore g is uniformly continuous on V.

Choose $\varepsilon_2 =$ ______. Then by the definition of uniform continuity there is $\delta > 0$ such that $x, y \in V$ and $|x-y| < \delta$ implies $|g(x) - g(y)| < \varepsilon_2$.

Now what's the partition?



Proof. Given $\epsilon > 0$, there is a finite collection of rectangles $R_1, \dots, R_M$ such that $Z \subset \bigcup_1^M R_m$ and the sum of the areas of the R_m's is less than ϵ. By subdividing these rectangles if necessary, we can assume that they have disjoint[2] interiors and form part of a grid obtained by partitioning some large rectangle R. Denoting this

Let $E = \{ (j,k) : R_{jk} \cap R_m \neq \emptyset \text{ for some } m \}$

where $R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$.

Need to make sure the partition is so small that the diagonal of each R_{jk} is less than δ.