4.18 Theorem. *Suppose g is a bounded function on the rectangle R. If the set of points in R at which g is discontinuous has zero content, then g is integrable on R.*

$R = [a,b] \times [c,d]$.

Since g is bounded there is a bound C such that

$$|g(x)| \leq C \quad \text{for every } x \in R.$$

Let $\varepsilon > 0$. Claim there is a partition

$$P = \{x_0, x_1, \dots, x_J ; y_0, y_1, \dots, y_K\}$$

Such that $|s_P g - S_P g| < \varepsilon$.

Choose $\varepsilon_1 = \dfrac{\varepsilon}{4C} > 0$.

Let $Z = \{x \in R : g \text{ is not cont. at } x\}$. Since Z has zero content there open rectangles R_m such that

$$\overline{Z} \subseteq \bigcup_{m=1}^{M} R_m \quad \text{and} \quad \sum_{m=1}^{M} |R_m| < \varepsilon_1$$

closed

complement of an open set is closed.

Note that $U = \bigcup_{m=1}^{M} R_m$ is open so $V = R \setminus U$ is closed.

union of open sets is open

closed..

Since the discontinuities of g are all in U then g is continuous on the set V. Since V is closed and bounded then V is compact

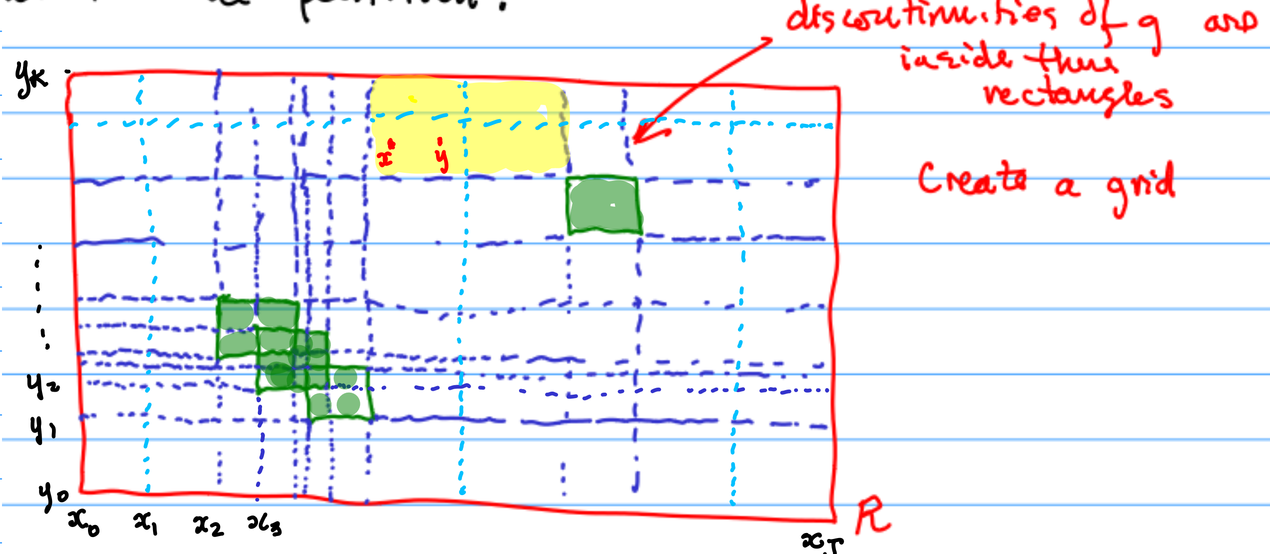
Therefore g is uniformly continuous on V.

Choose $\varepsilon_2 = \dfrac{\varepsilon}{2|R|} > 0$. Then by the definition of uniform continuity there is $\delta > 0$ such that $x, y \in V$ and $|x-y| < \delta$ implies $|g(x) - g(y)| < \varepsilon_2$.

Now find the partition ...

Such that $|s_P g - S_P g| < \varepsilon$.

What's the partition?



Further refine the partition so each rectangle has a diameter less than δ. Thus if $x, y \in R_{jk}$ then $|x-y| < \delta$.

$$R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$$

$$\boxed{\sum_{(j,k)\in E} |R_{jk}| \leq \sum_{m=1}^{M} |R_m| < \varepsilon_1}$$

where $E = \{(j,k) : R_{jk} \cap R_m \neq \emptyset \text{ for some } m\}$.

strictly less in the picture because the original rectangles overlapped.

$$S_p g = \sum_{j,k} M_{jk} |R_{jk}| \quad \text{where} \quad M_{jk} = \sup\{g(x) : x \in R_{jk}\}$$

$$s_p g = \sum_{j,k} m_{jk} |R_{jk}| \quad \text{where} \quad m_{jk} = \sup\{g(x) : x \in R_{jk}\}$$

$$S_p g = \sum_{(j,k)\in E} M_{jk} |R_{jk}| + \sum_{(j,k)\notin E} M_{jk} |R_{jk}|$$

$$s_p g = \sum_{(j,k)\in E} m_{jk} |R_{jk}| + \sum_{(j,k)\notin E} m_{jk} |R_{jk}|$$

$$S_p g - s_p g = \sum_{(j,k)\in E} (M_{jk} - m_{jk}) |R_{jk}| + \sum_{(j,k)\notin E} (M_{jk} - m_{jk}) |R_{jk}|$$

$$|M_{jk}| = |\sup\{g(x) : x \in R_{jk}\}| \leq \sup\{|g(x)| : x \in R_{jk}\} \leq C$$

since g is bounded

Similarly..

$$|m_{jk}| = |\inf\{g(x) : x \in R_{jk}\}| \leq C$$

By hypothesis

$|g(x)| \leq C$ for every $x \in R$.

$$-C \leq g(x) \leq C$$

$$-C \leq \inf\{g(x) : x \in R\} \leq \inf\{g(x) : x \in R_{jk}\} \leq C$$

$$C \geq -\inf\{g(x) : x \in R_{jk}\} \geq -C$$

Therefore $\left| -\inf\{g(x) : x \in R_{jk}\} \right| \leq C$

or equivalently $\left| \inf\{g(x) : x \in R_{jk}\} \right| \leq C$.

Summarize

$$|m_{jk}| \leq C \quad \text{and} \quad |M_{jk}| \leq C$$

Therefore

$$\left| \sum_{(j,k)\in E} (M_{jk} - m_{jk}) |R_{jk}| \right| \leq \sum_{(j,k)\in E} (|M_{jk}| + |m_{jk}|) |R_{jk}|$$

$$\leq \sum_{(j,k)\in E} 2C |R_{jk}| = 2C \sum_{(j,k)\in E} |R_{jk}| < 2C\varepsilon_1 = \frac{\varepsilon}{2}$$

since $\varepsilon_1 = \dfrac{\varepsilon}{4C}$

recall

$$\sum_{(j,k)\in E} |R_{jk}| \leq \sum_{m=1}^{M} |R_m| < \varepsilon_1$$

Now bound the other sum.

$$\left| \sum_{(j,k)\notin E} (M_{jk} - m_{jk}) |R_{jk}| \right|$$

$$= \sum_{(j,k)\notin E} |M_{jk} - m_{jk}| |R_{jk}|$$

$$M_{jk} = \sup\{g(x) : x \in R_{jk}\} = \max\{g(x) : x \in R_{jk}\} = g(b_{jk})$$

g is cont here so attains its maximum on the compact set R_{jk}

where the maximum is for some $b_{jk} \in R_{jk}$.

$$m_{jk} = \inf\{ g(x) : x \in R_{jk} \} = \min\{ g(x) : x \in R_{jk} \} = g(a_{jk})$$

for some $a_{jk} \in R_{jk}$.

since $a_{jk}, b_{jk} \in R_{jk}$ then diameter of R_{jk} being less than δ implies $|a_{jk} - b_{jk}| < \delta$ so $|g(b_{jk}) - g(a_{jk})| < \varepsilon_2$ unif cont.

$$\sum_{(j,k)\notin E} |M_{jk} - m_{jk}|\,|R_{jk}| = \sum_{(j,k)\notin E} |g(b_{jk}) - g(a_{jk})|\,|R_{jk}|$$

$$< \sum_{(j,k)\notin E} \varepsilon_2 |R_{jk}| \leq \varepsilon_2 |R| = \frac{\varepsilon}{2}$$

since $\varepsilon_2 = \dfrac{\varepsilon}{2|R|}$

It follows

$$|S_P g - s_P g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$