Dimensions $\mathbb{R}^n$ with $n > 2$ are similar.

The notion of "zero content" generalizes to $n$ dimensions in the obvious way: A bounded set $Z \subset \mathbb{R}^n$ has **zero content** if for any $\epsilon > 0$ there are rectangular boxes $R_1, \ldots, R_K$ whose total volume is less than $\epsilon$, such that $Z \subset \bigcup_1^K R_j$. The

cubes, or 4D boxes, etc...

Volume of $R_1 = |R_1|$

$R_1 = [a,b] \times [c,d] \times [e,f]$

$|R_1| = (b-a)(d-c)(f-e)$.

**4.24 Theorem** (The Mean Value Theorem for Integrals). *Let $S$ be a compact, connected, measurable susbset of $\mathbb{R}^n$, and let $f$ and $g$ be continuous functions on $S$ with $g \geq 0$. Then there is a point $c \in S$ such that*

$$\int \cdots \int_S f(\mathbf{x}) g(\mathbf{x})\, d^n\mathbf{x} = f(c) \int \cdots \int_S g(\mathbf{x})\, d^n\mathbf{x}.$$

$m = \inf\{f(x) : x \in S\} = f(a)$ $\quad$ $M = \sup\{f(x) : x \in S\} = f(b)$

Thus $\quad m \leq f(x) \leq M \quad$ for all $\quad x \in S$

Since $g(x) \geq 0$ then

$$g(x)\, m \leq g(x) f(x) \leq g(x) M \quad \text{for all } x \in S.$$

**4.17 Theorem.**

c. *If $f$ and $g$ are integrable on $S$ and $f(\mathbf{x}) \leq g(\mathbf{x})$ for $\mathbf{x} \in S$, then $\iint_S f\, dA \leq \iint_S g\, dA$.*

$S \subseteq \mathbb{R}^n \ldots$

constant

$$\int_S \cdots \int g(x)\, m\, d^n x \le \int_S \cdots \int g(x) f(x)\, d^n x \le \int_S \cdots \int g(x)\, M\, d^n x$$

constant comes out of integral because ....

*a. If $f_1$ and $f_2$ are integrable on the bounded set $S$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2$ is integrable on $S$, and*

$$\iint_S [c_1 f_1 + c_2 f_2]\, dA = c_1 \iint_S f_1\, dA + c_2 \iint_S f_2\, dA.$$

$$\underbrace{m \int_S \cdots \int g(x)\, d^n x}_{\text{same}} \le \int_S \cdots \int g(x) f(x)\, d^n x \le M \underbrace{\int_S \cdots \int g(x)\, d^n x}_{\text{same}}$$

divide by this ....

$$m \le \frac{\int_S \cdots \int g(x) f(x)\, d^n x}{\int_S \cdots \int g(x)\, d^n x} \le M$$

Therefore this between the max and min of $f$.

$$f(a) \le \underbrace{\frac{\int_S \cdots \int g(x) f(x)\, d^n x}{\int_S \cdots \int g(x)\, d^n x}}_{t} \le f(b)$$

Since $f$ is continuous the regular intermediate value theorem in $\mathbb{R}^n$ implies there is $c \in S$ such that

$$f(c) = \frac{\int \cdots \int_S g(x) f(x)\, d^n x}{\int \cdots \int_S g(x)\, d^n x},$$

*Proof.* Let $m$ and $M$ be the minimum and maximum values of $f$ on $S$, which exist since $S$ is compact. Since $g \geq 0$, we have $mg \leq fg \leq Mg$ on $S$, and hence

where is connected used in this proof?

$$m \int \cdots \int_S g(\mathbf{x})\, d^n\mathbf{x} \leq \int \cdots \int_S f(\mathbf{x}) g(\mathbf{x})\, d^n\mathbf{x} \leq M \int \cdots \int_S g(\mathbf{x})\, d^n\mathbf{x}.$$

Thus the quotient $(\int \cdots \int fg)/(\int \cdots \int g)$ lies between $m$ and $M$, so by the intermediate value theorem, it is equal to $f(\mathbf{a})$ for some $\mathbf{a} \in S$. $\square$

page 35 Corollary 1.27

**1.27 Corollary** (The Intermediate Value Theorem). *Suppose $f : S \to \mathbb{R}$ is continuous at every point of $S$ and $V \subset S$ is connected. If $\mathbf{a}, \mathbf{b} \in V$ and $f(\mathbf{a}) < t < f(\mathbf{b})$ or $f(\mathbf{b}) < t < f(\mathbf{a})$, there is a point $\mathbf{c} \in V$ such that $f(\mathbf{c}) = t$.*

Change of variables.

Let $g \in C^1([a,b])$ be 1-to-1

$f : \mathbb{R} \to \mathbb{R}$ is continuous...

$$\int_a^b f(g(u))\, g'(u)\, du = \int_{g(a)}^{g(b)} f(x)\, dx.$$

note if $g$ is 1-to-1 decreasing write $-\int_{g(b)}^{g(a)} f(x)\, dx$.

Before generalizing the u-substitution simplify thing be getting rid of the orientation of the top and bottom end points

$$R = [a,b] \qquad g(R) = \begin{cases} [g(a), g(b)] & \text{if } g \text{ is increasing} \\ [g(b), g(a)] & \text{if } g \text{ is decreasing.} \end{cases}$$

just writing $g(R)$ then don't have to consider whether $g$ is increasing or decreasing.

to get rid of the $-$ sign in the decreasing case

$$\int_{[a,b]} f(g(u))|g'(u)|\,du = \int_{g([a,b])} f(x)\,dx.$$

we generalize this to higher dimensions... It is possible to generalize the version with oriented intervals and that's call differential forms...

Orientation plays a role in the fundamental theorem of calculus in 1, 2, 3, ... and higher dimensions. Thats why you go around the boundary in a specified direction in, for example, Green's theorem..

$$\int_{[a,b]} f(g(u))|g'(u)|\,du = \int_{g([a,b])} f(x)\,dx.$$

Since $g$ is 1-to-1 then $g^{-1}(\underbrace{g([a,b])}_{S}) = [a,b]$

Let $S = g([a,b])$ then $[a,b] = g^{-1}(S)$

$$\int_S f(x)\,dx = \int_{g^{-1}(S)} f(g(u))\underbrace{|g'(u)|}_{|\det Dg(u)|}\,du.$$

in multiple dimensions

We'll build up to the multi-dimensional case by treating the case

$$g(u) = Au \quad \text{where } A \in \mathbb{R}^{n \times n}$$

to begin with.