① One more homework

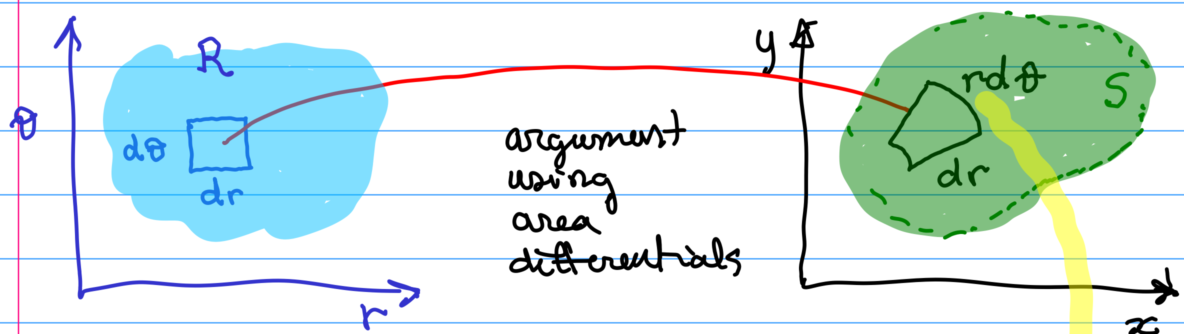
② Then a sequence of 5 short quiz until the final.

The final exams will be held in person at the time listed in the standard schedule of final exams for this section. Namely, the final exam is Wednesday, May 15 from 8:00-10:00am in PE103.

Change of variables like in Math 283

Polar coordinates

$$G(r,\theta) = (r\cos\theta,\ r\sin\theta)$$



Conclusion

$$\iint_S f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$G(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

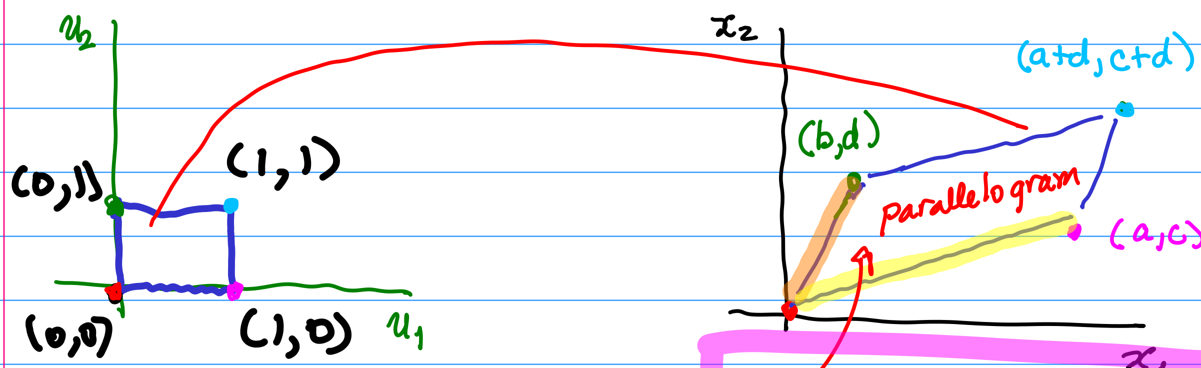
$$DG(r,\theta) = \begin{bmatrix} \dfrac{\partial r\cos\theta}{\partial r} & \dfrac{\partial r\cos\theta}{\partial \theta} \\[2ex] \dfrac{\partial r\sin\theta}{\partial r} & \dfrac{\partial r\sin\theta}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\det DG(r,\theta) = \cos\theta \cdot r\cos\theta - \sin\theta(-r\sin\theta) = r$$

$$|\det DG(r,\theta)| = |r| = r$$

$n=2 \qquad A \in \mathbb{R}^{2\times 2}$

$$G(u) = Au = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

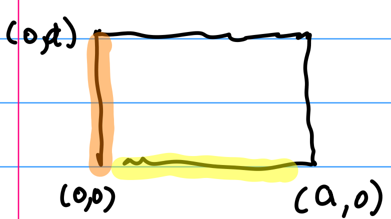


$\boxed{\text{area} = \det A = ad - bc}$

THEOREM 9

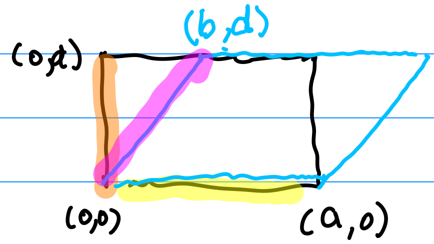
If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Proof: if A is diagonal then $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$



$$\left.\begin{aligned} \text{area} &= da \\ \det A &= ad \end{aligned}\right\} \text{ the same...}$$

Shear this



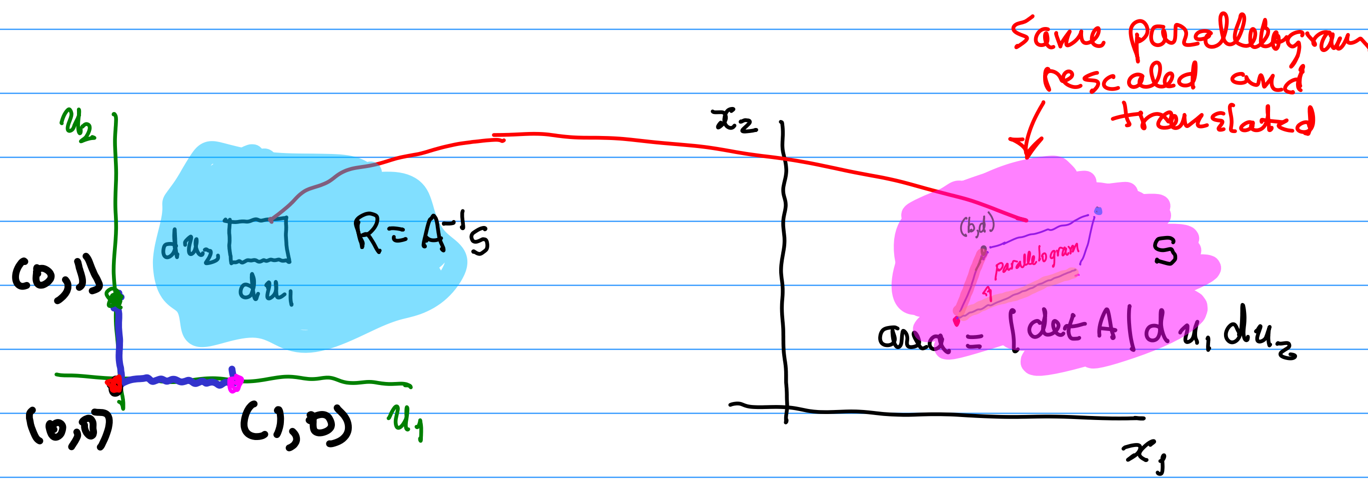
area stays the same

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \qquad c_2 \leftarrow c_2 + \frac{b}{a} c_1$$

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

determinant stays the same.

In the end, the area $= \det|A|$ because neither changes when performing elimination steps on the columns...



Heuristically we now have

If. $G(x) = Ax$ then $DG(x) = A$ and

$$\int \cdots \int_S f(x)\, d^n x \approx \int \cdots \int_{A^{-1}(S)} f(Au)\, |\det A|\, d^n u$$

4.37 Theorem. *Let A be an invertible $n \times n$ matrix, and let $\mathbf{G}(\mathbf{u}) = A\mathbf{u}$ be the corresponding linear transformation of $\mathbb{R}^n$. Suppose S is a measurable region in $\mathbb{R}^n$ and f is an integrable function on S. Then $\mathbf{G}^{-1}(S) = \{A^{-1}\mathbf{x} : \mathbf{x} \in S\}$ is measurable and $f \circ \mathbf{G}$ is integrable on $\mathbf{G}^{-1}(S)$, and* $R = G^{-1}(S)$

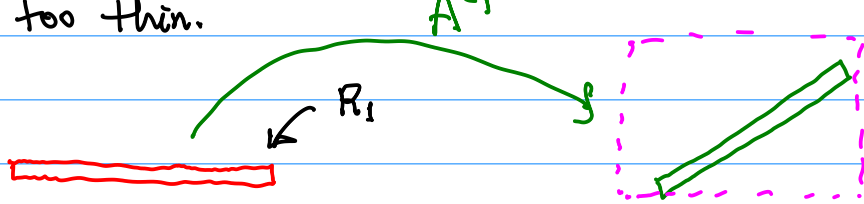
$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = |\det A| \int \cdots \int_{\mathbf{G}^{-1}(S)} f(A\mathbf{u})\, d^n\mathbf{u}. \tag{4.38}$$

Let $\varepsilon > 0$ arbitrary...

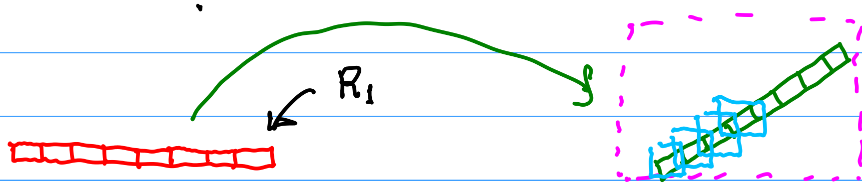
Since S is measurable the for any $\varepsilon_1 = \boxed{}$ there are rectangles R_m such that

$$\partial S \subseteq \bigcup_{m=1}^{M} R_m \quad \text{and} \quad \sum_{m=1}^{M} |R_m| < \varepsilon_1$$

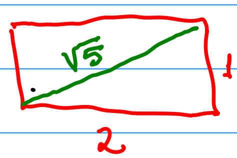
Things go badly if the rectangles R_m are too long and too thin.



To avoid this problem, lets assume the rectangles R_m have been subdivides so the length is less than twice the width



Worst case



$$(\text{diam } R_1)^2 = 5 \qquad |R_1| = 2$$

$$\frac{2}{5}(\text{diam } R_m)^2 \leq |R_m| \leq (\text{diam } R_m)^2 \qquad \text{for all } m$$

Let Q_m be rectangle

$$\partial G^{-1}(S) = \partial A^{-1} S = A^{-1} \partial S \subseteq A^{-1} \bigcup_{m=1}^{M} R_m = \bigcup_{m=1}^{M} \underbrace{A^{-1} R_m}_{\text{not rectangles...}}$$

Now estimate the size of the rectangles Q_m that contain the parallelograms $A^{-1} R_m$.

Next time...