For next Wednesday please read Chapter 5.1 about line integrals. Monday is a holiday!

HW5 due Nov 15

Turn in (page 167) 4.2#7

~~Practice (page~~ 158) 4.1#6, (page 167) 4.2#3,4,5 (page 176) 4.3#5abc

... new homework ...

4.37 Theorem. *Let A be an invertible $n \times n$ matrix, and let $\mathbf{G}(\mathbf{u}) = A\mathbf{u}$ be the corresponding linear transformation of $\mathbb{R}^n$. Suppose S is a measurable region in $\mathbb{R}^n$ and f is an integrable function on S. Then $\mathbf{G}^{-1}(S) = \{A^{-1}\mathbf{x} : \mathbf{x} \in S\}$ is measurable and $f \circ \mathbf{G}$ is integrable on $\mathbf{G}^{-1}(S)$, and*

$R = G^{-1}(S)$

$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = |\det A| \int \cdots \int_{\mathbf{G}^{-1}(S)} f(A\mathbf{u})\, d^n\mathbf{u}. \tag{4.38}$$

Assume S is Jordan measurable, Thus ∂S has zero content.

Try to show $G^{-1}(S)$ is Jordan measurable.

Need to show $\partial G^{-1}(S)$ has zero content.

Let $\varepsilon > 0$ arbitrary...

Since S is measurable the for any $\varepsilon_1 = \dfrac{\boxed{\varepsilon}}{\|A^{-1}\|^2 \frac{5}{2}}$

there are rectangles R_m such that

$$\partial S \subseteq \bigcup_{m=1}^{M} R_m \quad \text{and} \quad \sum_{m=1}^{M} |R_m| < \varepsilon_1$$

assume

$$\frac{2}{5} (\operatorname{diam} R_m)^2 \leq |R_m| \leq (\operatorname{diam} R_m)^2 \quad \text{for all } m$$

$$(\operatorname{diam} R_m)^2 \leq \frac{5}{2} |R_m|$$

$$\partial G^{-1}(S) = \partial A^{-1}S = A^{-1}\partial S \subseteq A^{-1}\bigcup_{m=1}^{M} R_m = \bigcup_{m=1}^{M} A^{-1}R_m$$

no rectangles.

$$\operatorname{diam} A^{-1}R_n = \max\{\|A^{-1}x - A^{-1}y\| : x,y \in R_m\}.$$

Let $x,y \in R_m$

matrix norm is finite.

$$\|A^{-1}x - A^{-1}y\| = \|A^{-1}(x-y)\| \leq \|A^{-1}\|\,\|x-y\|$$

Therefore

$$\operatorname{diam} A^{-1}R_n \leq \|A^{-1}\| \operatorname{diam} R_m$$

$$A^{-1}R_m \subseteq Q_m = [a_m, b_m]\times[c_m, d_m] \quad \text{where} \quad |b_m - a_m| = \operatorname{diam} A^{-1}R_m$$

$$|d_m - c_m| = \operatorname{diam} A^{-1}R_m$$

not a rectangle

rectangle with side the length equal the diam $A^{-1}R_m$

Then

$$\partial G^{-1}(S) \subseteq \bigcup_{m=1}^{M} Q_m \quad \text{where} \quad |Q_m| = (\operatorname{diam} A^{-1}R_m)^2$$

need to show to total area of the Q_m's is small

$$|Q_m| = (\operatorname{diam} A^{-1}R_m)^2 \leq \|A^{-1}\|^2(\operatorname{diam} R_m)^2$$

$$\leq \|A^{-1}\|^2 \tfrac{5}{2}|R_m|$$

since

$$(\operatorname{diam} R_m)^2 \leq \tfrac{5}{2}|R_m|$$

$$\sum_{m=1}^{M}|Q_m| \leq \|A^{-1}\|^2 \tfrac{5}{2}\sum_{m=1}^{M}|R_m| < \|A^{-1}\|^2 \tfrac{5}{2}\varepsilon_1 = \cancel{\|A^{-1}\|^2 \tfrac{5}{2}}\frac{\varepsilon}{\cancel{\|A^{-1}\|^2 \tfrac{5}{2}}} = \varepsilon$$

Idea decompose $\underbrace{A = E_1 E_2 E_3 \cdots E_N}$ where E_i are matrices corresponding to elementary row operations. This can be done because A^{-1} exists.

1. Multiply the kth component by a nonzero number c, rescaling $c \neq 0$ $r_k \leftarrow c\, r_k$

   affect only one dimension

2. Add a multiple of the jth component to the kth component, elimination step $r_k \leftarrow r_k + c\, r_j$

3. Interchange the jth and kth components: row swap $r_k \leftrightarrow r_j$

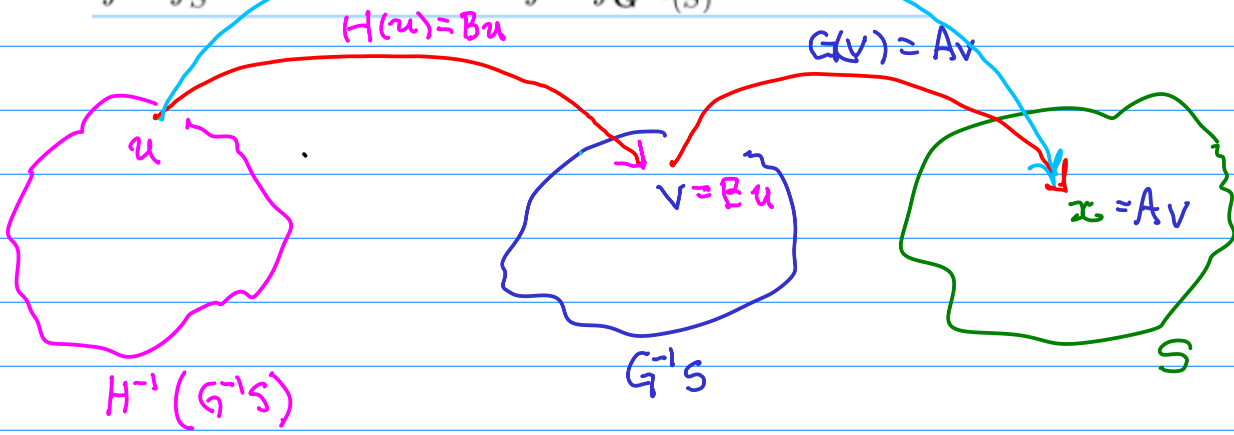
   actually two here

each E_i corresponds to one of these operations.

Need to show can build the matrix back up by mult the elementary matrices back together in a way that preserves

$G \circ H(u) = (AB)u, \ldots$

$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = |\det A| \int \cdots \int_{\mathbf{G}^{-1}(S)} f(A\mathbf{u})\, d^n\mathbf{u}.$$



$$G \circ H(u) = G(H(u)) = G(Bu) = ABu$$

Suppose the formula holds for $\underline{\underline{G}}$ and H seperately

$$\int \cdots \int_S f(x)\, d^n x = |\det A| \int \cdots \int_{G^{-1}(S)} f(Av)\, d^n v$$

$$\int \cdots \int_{G^{-1}(S)} f(Av)\, d^n v = |\det B| \int \cdots \int_{H^{-1}(G^{-1}(S))} f(A(Bu))\, d^n u$$

Therefore

$$\int \cdots \int_{S} f(x)\, d^n x = |\det A|\,|\det B| \int \cdots \int_{H^{-1}(G^{-1}(S))} f(A(Bu))\, d^n u$$

Now simplify

$$|\det A|\,|\det B| = |\det AB|$$

$$H^{-1}(G^{-1}(S)) = B^{-1}A^{-1}S = (AB)^{-1}(S) = (G \circ H)^{-1}(S)$$

$$\int \cdots \int_{S} f(x)\, d^n x = |\det AB| \int \cdots \int_{(G \circ H)^{-1}(S)} f(A(Bu))\, d^n u$$

This is the change of variables formula for the composition formed by multiplying the matrices together.

What's left is to prove the formula for the elementary matrices. This can be done by first writing the multidimensional integral into a series of iterated integrals and working one dimension at a time. This work because the elementary row operation are so simple that only one dimension needs to be considered to see what they do...

three assumptions of integrability

B.9 Theorem. *Let $R = [a,b] \times [c,d]$, and let f be an integrable function on R. Suppose that, for each $y \in [c,d]$, the function f_y defined by $f_y(x) = f(x,y)$ is integrable on $[a,b]$, and the function $g(y) = \int_a^b f(x,y)\,dx$ is integrable on $[c,d]$. Then*

$$\iint_R f\,dA = \int_c^d \left[\underbrace{\int_a^b f(x,y)\,dx}_{g.} \right] dy.$$

Equivalently $\iint_R f\,dA = \int_a^d g(y)\,dy$

Proof is an $\varepsilon/3$ argument based on all of the assumptions of integrability.

We'll discuss the proof on Wednesday. Don't forget to read Chapter 5.1 over the long weekend.