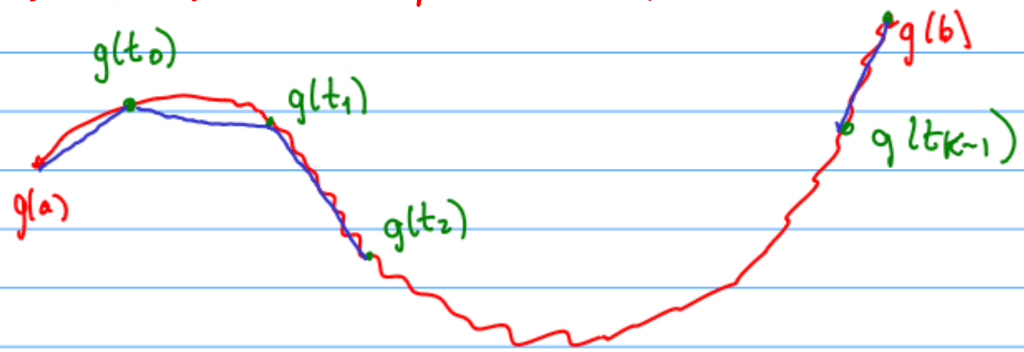
Let C be a curve and $g:[a,b] \to C$ be a parameterization of the curve that's continuous and one to one.

$a = t_0 < t_1 < t_2 \cdots < t_K = b$

$P = \{t_0, \dots, t_K\}$ is a partition of $[a,b]$.



$$L_P(C) = \sum_{k=1}^{K} |\underbrace{g(t_k) - g(t_{k-1})}|$$ ← these are all too small

$$\underbrace{L(C)} = \sup\{\underbrace{L_P(C)} : P \text{ is a partition of } [a,b]\}$$

Supremum $L(C)$ always exists in $\mathbb{R} \cup \{\infty\}$, but is it finite? For what curves is it finite?

Does the supremum exist in $\mathbb{R}$.

- If so then the curve is rectifiable.
- Otherwise not rectifiable...

Note that the definition of $L(C)$ ultimately depends on g.
If one uses a different parameterization of C does $L(C)$ change? No

$g:[a,b] \to C$ g is continuous and one to one
and also

5.11 Theorem. *With notation as above, if* $\mathbf{g}$ *is of class* C^1, *then* C *is rectifiable, and*

$$L(C) = \int_a^b |\mathbf{g}'(t)|\,dt.$$

? $\boxed{\checkmark}$

Let $P = \{t_0, t_1, \dots, t_J\}$ be a partition of $[a,b]$.

Then

$$L_P(C) = \sum_{j=1}^{J} \underbrace{|g(t_j) - g(t_{j-1})|}_{\text{write this difference using Fundamental Theorem of Calculus}} = \sum_{j=1}^{J} \left| \int_{t_{j-1}}^{t_j} g'(t)\,dt \right|$$

$$\leq \underbrace{\sum_{j=1}^{J} \int_{t_{j-1}}^{t_j}}_{\text{sum of a bunch of integrals of the same thing.}} |g'(t)|\,dt = \int_a^b |g'(t)|\,dt$$

Thus we have an upper bound

$$L_P(C) \leq \int_a^b |g'(t)|\,dt$$

Thus

$$\{ L_P(C) : P \text{ is a partition of } [a,b] \}$$

is bounded above and so by the completeness axiom

$$\sup \{ L_P(C) : P \text{ is a partition of } [a,b] \} =$$

is a finite real number. Thus $L(C) \leq \int_a^b |g'(t)|\,dt$.

Examples of curves that aren't rectifiable can be found in the study of fractals...

Still need to show $L(C) = \int_a^b |\mathbf{g}'(t)|\, dt.$

Lets define $C_r^s = g([r,s])$ for $[r,s] \subseteq [a,b]$.

part of the curve.

define $\varphi(s) = L(C_a^s)$ for all $s \in [a,b]$.

(note $\varphi(a)=0$).

Claim φ is differentiable.

Case $h>0$. Consider $\dfrac{\varphi(s+h)-\varphi(s)}{h}$ and try to take limits.

$$\varphi(s+h)-\varphi(s) = L(C_a^{s+h}) - L(C_a^s)$$

Note

$$L(C_a^s) + L(C_s^{s+h}) = L(C_a^{s+h})$$

Thus $\varphi(s+h)-\varphi(s) = L(C_s^{s+h}) \geq$

$L(C_s^{s+h}) = \sup\{L_p(C_s^{s+h})$ where p is a partition of $[s,s+h]\}$

Let $P=\{s, s+h\}$ simplest partition possible

$$L_p(C_s^{s+h}) = |g(s+h)-g(s)|$$

Thus $|g(s+h)-g(s)| \leq \varphi(s+h)-\varphi(s)$.

also

average value of $|g'(t)|$ on $[s, s+h]$

$$L(C_s^{s+h}) \leq \int_s^{s+h} \underbrace{|g'(t)|}_{\text{continuous function}}dt = h \underbrace{\boxed{\frac{\int_s^{s+h} |g'(t)|dt}{h}}}_{\text{mean value theorem for integrals.}} = h\,|g'(\sigma)| \quad \text{where } \sigma \in [s, s+h]$$

Consequently

$$\frac{|g(s+h)-g(s)|}{h} \leq \frac{\varphi(s+h)-\varphi(s)}{h} = \frac{L(C_s^{s+h})}{h} \leq \frac{h\,|g'(\sigma)|}{h}$$

where $\sigma \in [s, s+h]$

$$\frac{|g(s+h)-g(s)|}{h} \leq \frac{\varphi(s+h)-\varphi(s)}{h} \leq |g'(\sigma)|$$

as $h \to 0^+$ since g is differentiable. $|g'(s)|$

as $h \to 0^+$ since g' is continuous. $|g'(s)|$

Consequently

$$\lim_{h \to 0^+} \frac{\varphi(s+h)-\varphi(s)}{h} = |g'(s)|$$

common value on left and right.

Now since φ' exists we use fund. theorem again

$$L(C) = L(C_a^b) = \varphi(b) - \varphi(a) = \int_a^b \varphi'(t)dt = \int_a^b |g'(t)|dt$$

Line integrals...

Scalar functions

$f : D \to \mathbb{R}$ where $C \subseteq D$

let g be a one-to-one C^1 parameterization of C

$g : [a,b] \to C$

$$\underbrace{\int_C f\, ds = \int_{[a,b]} f(g(t))\, |g'(t)|\, dt}_{\text{this is the definition of } \int_C f ds.}$$

Vector functions

$$\underbrace{\int_C F \cdot dx = \int_{[a,b]} F(g(t)) \cdot g'(t)\, dt}_{\text{definition of } \int_C F\cdot dx.}$$

Claim $\int_C f\, ds$ doesn't depend on the parameterization.

Let $\varphi : [c,d] \to [a,b]$ be a change of parameterization

$h : [c,d] \to C$ ← new parameterization.

$h(u) = g(\varphi(u))$

note if we start with h then can solve for φ using inverse funct theorem and we get that φ is C^1.

Therefore $$\int_I f(x)\, dx = \int_{g^{-1}(I)} f(g(u))|g'(u)|\, du.$$

↓ change of vbls.

$$\int_{[a,b]} f(g(t))\, |g'(t)|\, dt = \int_{\varphi^{-1}([a,b])} f(\underline{g(\varphi}(u)))\, |g'(\varphi(u))|\, |\varphi'(u)|\, du$$

$$= \int_{[c,d]} f(h(u)) \, \Big| \underbrace{g'(\varphi(u))\, \varphi'(u)}_{\text{chain rule}} \Big| \, du$$

$$= \int_{[c,d]} f(h(u)) \, \big| (g \circ \varphi)'(u) \big| \, du = \int_{[c,d]} f(h(u)) \, |h'(u)| \, du$$

Changing the parameterization in

$$\int_C F \cdot dx$$

depends on the orientation of the parameterization.

Next time...