Leading up to the final exam there will be a quiz every day. I'll provide clues what is on the quiz a day or two ahead of each quiz. Look on the weekend for information about the quiz on Monday.

S Compact and $S = \overline{S^{int}}$

5.12 Theorem (Green's Theorem)**.** *Suppose S is a regular region in $\mathbb{R}^2$ with ~~piece-wise~~ smooth boundary ∂S. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on $\overline{S}$. Then*

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA. \tag{5.13}$$

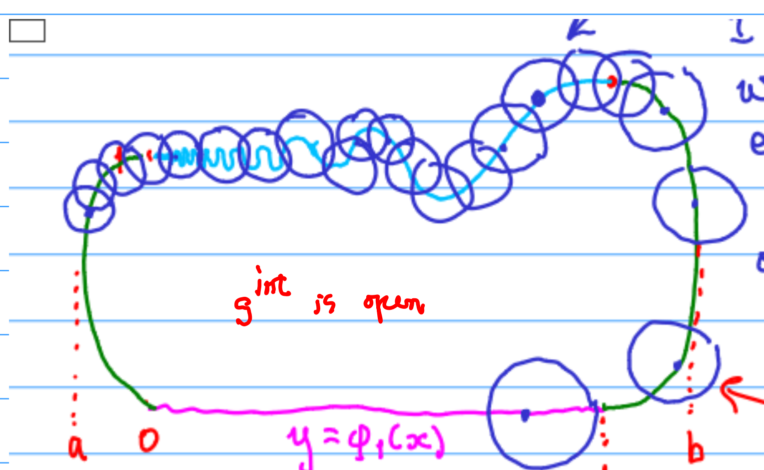
In the more common notation, if we set $\mathbf{F} = (P, Q)$ and $\mathbf{x} = (x, y)$,

$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA. \tag{5.14}$$

We've done simple regions, that is both x-simple and y-simple at the same time.

We've done any region that can be decomposed into simple regions...

We'll work on smooth boundaries



$S \subseteq \bigcup_{j=1}^{N} U_j$

$U_1 = S^{int}$

U_{j+1} are the open sets which cover the boundary.

So create a grid of rectangles such that for each rectangle R_m that intersects S there is an open set $U_{j(m)}$ where $R_m \subseteq U_{j(m)}$.

made rectangles bigger so they over lap. But still $\tilde{R}_m \subseteq U_{j(m)}$

And the rectangles that don't intersect S still don't.

We used these bigger rectanges to create the partition of unity... φ_m.

$$\boxed{\sum_{m=1}^{M} \varphi_m(x)} = 1 \quad \text{for } x \in S$$

extend φ_m so they are defined on all of $\mathbb{R}^2$ and equal $\varphi_m(x) = 0$ for $x \in \tilde{R}_m$

We want to prove that

$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_{\partial S} P dx + Q dy = \sum_{m=1}^{M} \int_{\partial S} P_m dx + Q_m dy$$

where $P_m(x,y) = P(x,y)\,\varphi_m(x,y)$

and $Q_m(x,y) = Q(x,y)\,\varphi_m(x,y)$

also

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \sum_{m=1}^{M} \iint_S \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dA$$

☑ Claim $\displaystyle\int_{\partial S} P_m dx + Q_m dy = \iint_S \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dA$

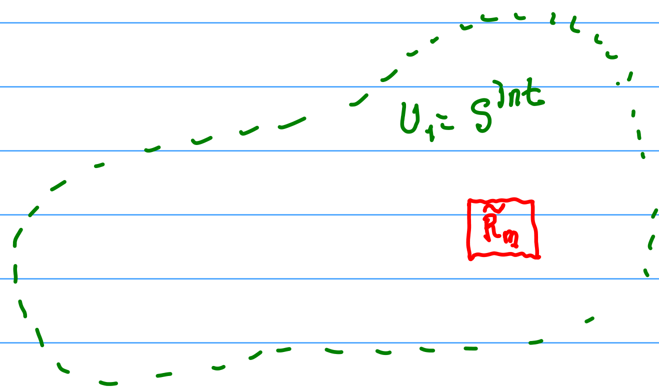
To deal with more complicated regions S we break P and Q apart into simpler functions.

Fix m what do we have? $\tilde{R}_m \subseteq U_{j(m)}$

Two cases $j(m)=1$ so $U_{j(m)} = U_1 = S^{int}$

other $j(m)>1$ so $U_{j(m)}$ is on the boundary...

Case $j(m)=1$:

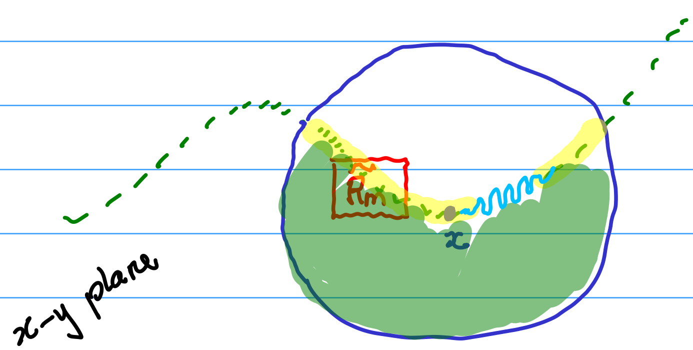


in this case
$\varphi_m(x) = 0$ on $\partial \tilde{R}_m$
$\varphi_m(x) = 0$ for $x \notin \tilde{R}_m$
consequently
$\varphi_m(x) = 0$ on ∂S

$$0 = \int_{\partial S} P_m\,dx + Q_m\,dy = \iint_S \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dA = 0$$

So $0=0$ and the claim is verified.

Case $j(m)>1$. $x \in \partial S$



$\partial S \cap U_{j(m)}$ is a graph of either $\boxed{y = f(x)}$ or $x = g(y)$

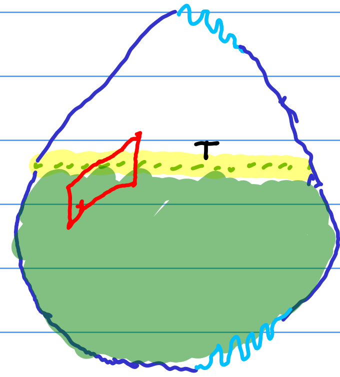
Change of variables

$u = x \qquad v = y - f(x)$

$x = u \qquad y = v + f(x) = v + f(u)$

$G(u,v) = (u, v + f(u)) \qquad G^{-1}(x,y) = (x, y - f(x))$

u-v $\qquad$ v=0



this is a simple region?, No

but I've moved the oscillations to the side of the curve where the function P_m and Q_m are zero

$$\int_{\partial S} P_m\, dx + Q_m\, dy = \int_{\partial S \cap U_{j(m)}} P_m\, dx + Q_m\, dy =$$

$$\tilde{Q}(u,v) = Q_m \circ G(u,v)$$

$$= \int_T \underbrace{P_m \circ G(u,v)}_{\tilde{P}(u,v) = P_m \circ G(u,v)}\, du + \overbrace{Q_m \circ G(u,v)}^{\tilde{Q}(u,v)}\,(dv + f'(u)\, du)$$

$G(u,v) = (u, v + f(u)) \qquad G^{-1}(x,y) = (x, y - f(x)) \qquad T = G^{-1}(\partial S \cap U_{j(m)})$

$u = x \qquad v = y - f(x)$

$x = u \qquad y = v + f(x) = v + f(u)$

$dx = du \qquad dy = dv + f'(u)\, du$

$$\ldots = \int_T \tilde{P}\, du + \tilde{Q}\, dv + \tilde{Q} f'(u)\, du = \int_T (\tilde{P} + \tilde{Q} f'(u))\, du + \tilde{Q}\, dv$$

$$\iint_S \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dA = \iint_{S \cap U_{m(j)}} \left(\frac{\partial Q_m}{\partial x} - \frac{\partial P_m}{\partial y}\right) dx\, dy$$

$$\ldots = \iint_{G^{-1}(S \cap U_{m(j)})} \left(\frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y}\right) |\det DG(u)|\, du\, dv$$

$u = x \qquad v = y - f(x)$

$$\frac{\partial \tilde{Q}}{\partial x} = \frac{\partial \tilde{Q}}{\partial v}\frac{dv}{dx} + \frac{\partial \tilde{Q}}{\partial u}\frac{du}{dx} = \frac{\partial \tilde{Q}}{\partial v}\left(-f'(x)\right) + \frac{\partial \tilde{Q}}{\partial u}$$

$$\frac{\partial \tilde{P}}{\partial y} = \frac{\partial \tilde{P}}{\partial v}\frac{dv}{dy} + \frac{\partial \tilde{P}}{\partial u}\frac{du}{dy} = \frac{\partial \tilde{P}}{\partial v} + 0 = \frac{\partial \tilde{P}}{\partial v}$$