HW1 due Sept 13

Turn in (page 33) 1.6#4

Practice (page 29) 1.5#5, 1.5#7, (page 33) 1.6#1ab, 1.6#6, (page 38) 1.7#6, 1.7#8

§1.6# **5.** Define a sequence $\{x_k\}$ recursively by $x_1 = \sqrt{2}$, $x_{k+1} = \sqrt{2 + x_k}$. Show by induction that (a) $x_k < 2$ and (b) $x_k < x_{k+1}$ for all k. Then show that $\lim x_k$ exists and evaluate it.

(a). Since $x_1^2 = (\sqrt{2})^2 = 2 < 4 = 2^2$ it follows $0 < x_1 < 2$.

For induction suppose $0 < x_k < 2$, then

$$x_{k+1} = \sqrt{2 + x_k} < \sqrt{2+2} = \sqrt{4} = 2$$

and

$$x_{k+1} = \sqrt{2 + x_k} > \sqrt{2} > 0.$$

Thus $0 < x_{k+1} < 2$ which completes the induction. Therefore we have $0 < x_k < 2$ for all $k \in \mathbb{N}$.

(b). Since $x_1 = \sqrt{2}$ then $x_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2} = x_1$ so $x_1 < x_2$.

For induction suppose $x_k < x_{k+1}$, then

$$x_{k+2} = \sqrt{2 + x_{k+1}} > \sqrt{2 + x_k} = x_{k+1},$$ so $x_{k+1} < x_{k+2}$ which completes the induction.

Therefore $x_k < x_{k+1}$ for all $k \in \mathbb{N}$.

It follows that x_k is a bounded monotone increasing sequence and consequently by the monotone convergence theorem the limit of x_k as $k \to \infty$ exists, Let $\alpha \in \mathbb{R}$ be the limit so $x_k \to \alpha$ as $k \to \infty$.

Now,

$$\alpha = \lim_{k\to\infty} x_k = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} \sqrt{2+x_k} = \sqrt{2+\alpha}$$

implies

$$\alpha^2 = 2 + \alpha \qquad \text{or} \qquad \alpha^2 - \alpha - 2 = 0$$

Setting $a=1$, $b=-1$ and $c=-2$ yields that

$$\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1\pm 3}{2} = -1 \text{ or } 2$$

Since $\alpha > 0$ it follows that $x_k \to 2$ as $k \to \infty$.

§1.5# **7.** Let $\{\mathbf{x}_k\}$ be a sequence in $\mathbb{R}^n$ and $\mathbf{x}$ a point in $\mathbb{R}^n$. Show that some subsequence of $\{\mathbf{x}_k\}$ converges to $\mathbf{x}$ if and only if every ball centered at $\mathbf{x}$ contains $\mathbf{x}_k$ for infinitely many values of k.

"$\Leftarrow$" Consider the sequence of balls given by $B_n = B(\frac{1}{n}, x)$.

Since $\{x_k\} \cap B_1 \neq \emptyset$ there is k_1 such that $x_{k_1} \in B_1$.

Since $\{k : x_k \in B_2\}$ is infinite then $\{x_k : k > k_1\} \cap B_2 \neq \emptyset$.
Therefore, there is $k_2 > k_1$ such that $x_{k_2} \in B_2$.

Given $x_{k_n} \in B_n$ note that $\{k : x_k \in B_{n+1}\}$ being infinite implies $\{x_k : k > k_n\} \cap B_{n+1} \neq \emptyset$. Therefore there is $k_{n+1} > k_n$ such that $x_{k_{n+1}} \in B_{n+1}$

By induction we obtain a subsequence x_{k_n} such that $x_{k_n} \in B_{k_n}$ for every $n \in \mathbb{N}$.

Claim $x_{k_n} \to x$ as $n \to \infty$.

Let $\varepsilon > 0$. Then there is N so large $\frac{1}{N} < \varepsilon$. Since for any subsequence $k_n \geq n$ we have $\frac{1}{k_n} \leq \frac{1}{n} \leq \frac{1}{N}$ for $n \geq N$.

Now since $x_{k_n} \in B_{k_n} = B(\frac{1}{k_n}, x)$ then

$$|x_{k_n} - x| < \frac{1}{k_n} \leq \frac{1}{N} < \varepsilon \quad \text{for all} \quad n \geq N.$$

This means $x_{k_n} \to x$ as $n \to \infty$.

"$\Rightarrow$" Suppose there is a subsequence such that $x_{k_n} \to x$ as $n \to \infty$.

Consider any ball $B = B(r, x)$ where $r > 0$.

Since $x_{k_n} \to x$ by definition of convergence there is N large enough such that

$$|x_{k_n} - x| < r \quad \text{for all} \quad n \geq N.$$

But then $x_{k_n} \in B$ for all $n \geq N$ and in particular

$$\{k_n : n \geq N\} \subseteq \{k : x_k \in B\}.$$

Since the set on the left is infinite then so is the one on the right.

§1.6#

1. Give an example of

a. a closed set $S \subset \mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(S)$ is not closed;

b. an open set $U \subset \mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(U)$ is not open.

(a) Let $S = [0, \infty)$ and $f(x) = \frac{1}{1+x}$.

Then S is closed and

$f([0,\infty)) = (0, 1]$ is not closed.

(b) Let $S = (-1, 1)$ and $f(x) = 2$

Then S is open and

$f((-1,1)) = \{2\}$ is not open.

4. Suppose $S \subset \mathbb{R}^n$ is compact, $f : S \to \mathbb{R}$ is continuous, and $f(\mathbf{x}) > 0$ for every $\mathbf{x} \in S$. Show that there is a number $c > 0$ such that $f(\mathbf{x}) \geq c$ for every $\mathbf{x} \in S$.

For contradiction, if not then there is a sequence $x_k \in S$ such that $f(x_k) < \frac{1}{k}$ for $k \in \mathbb{N}$.

Since S is bounded then the Bolzano-Weierstrass theorem implies there is a convergent subsequence $x_{k_n} \to x$ and S closed implies $x \in S$.

By hypothesis $x \in S$ implies $f(x) > 0$.

By continuity $f(x_{k_n}) \to f(x)$ as $n \to \infty$.

Let $\varepsilon = f(x)$ and choose N_1 large enough so that

$$|f(x_{k_n}) - f(x)| < \varepsilon/2 \quad \text{for} \quad n \geq N_1$$

and N_2 large enough that

$$\frac{1}{k_n} < \frac{\varepsilon}{2} \quad \text{for} \quad n \geq N_2$$

Let $N = \max(N_1, N_2)$. It follows that

$$\varepsilon = f(x) \leq |f(x) - f(x_{k_n})| + f(x_{k_n}) < \frac{\varepsilon}{2} + \frac{1}{k_n} < \varepsilon$$

which is a contradiction.

Therefore there is $c > 0$ such that $f(x) \geq c$ for every $x \in S$.

6. The **distance** between two sets $U, V \subset \mathbb{R}^n$ is defined to be

$$d(U, V) = \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{x} \in U,\ \mathbf{y} \in V\}.$$

a. Show that $d(U, V) = 0$ if either of the sets U, V contains a point in the closure of the other one.
b. Show that if U is compact, V is closed, and $U \cap V = \varnothing$, then $d(U, V) > 0$.
c. Give an example of two closed sets U and V in $\mathbb{R}^2$ that have no point in common but satisfy $d(U, V) = 0$.

(a) If $U \cap \overline{V} \neq \emptyset$ then there is $x \in U$ and $y_n \in V$ such that $y_n \to x$ as $n \to \infty$. It follows that

$$0 \leq d(U,V) = \inf\{|x-y| : x \in U, y \in V\} \leq |x - y_n| \quad \text{for all } n.$$

Since $y_n \to x$ it follows that $|x - y_n| \to 0$.

Therefore $d(U,V) = 0$.

The case $\overline{U} \cap V \neq \emptyset$ is identical because

$$d(U,V) = \inf\{|x-y| : x \in U, y \in V\} = \inf\{|x-y| : x \in V, y \in U\} = d(V,U).$$

(b) Let U be compact, V closed and $U \cap V = \emptyset$.
Suppose for contradiction that $d(U,V) = 0$. Then there would be $x_k \in U$ and $y_k \in V$ such that $|x_k - y_k| \to 0$ as $k \to \infty$.

Since U is bounded, then by the Bolzano-Weierstrass theorem there is a subsequence $x_{k_n} \to x$ as $n \to \infty$. Since U is also closed then $x \in U$.

Now $|y_{k_n} - x| \le |x_{k_n} - x| + |x_{k_n} - y_{k_n}| \to 0$ implies that also $y_{k_n} \to x$. Since V is closed this would mean that $x \in V$. But then $U \cap V \neq \emptyset$ which contradicts the hypothesis $U \cap V = \emptyset$. It follows that $d(U,V) > 0$.

(c) Let $U = \{(x, \frac{1}{x}) : x \geq 1\}$ and $V = \{(x, -\frac{1}{x}) : x \geq 1\}$.

Claim U is closed. To show this we need to prove every convergent sequence in U has its limit also in U.

Suppose $u_n \in U$ with $u_n \to u$ for some $u \in \mathbb{R}^2$. Write $u_n = (x_n, \frac{1}{x_n})$ and $u = (x, y)$. Then $u_n \to u$ implies that $x_n \to x$ since $x_n \geq 1$ then $x \geq 1$.

Let $f : [1, \infty) \to \mathbb{R}$ be given by $f(t) = \frac{1}{t}$. Since f is continuous on $[1, \infty)$ then $f(x_n) \to f(x)$ as $n \to \infty$. This implies $u_n = (x_n, \frac{1}{x_n}) \to (x, \frac{1}{x})$. Uniqueness of limits now implies $y = \frac{1}{x}$ and so $u = (x, y) \in U$.

Therefore U is closed. Similar arguments imply V is closed.

Clearly $U \cap V = \emptyset$. To see $d(U,V) = 0$ note that

$$\inf\{|x - y| : x \in U, y \in V\} \le \left|(k, \tfrac{1}{k}) - (k, -\tfrac{1}{k})\right| = \frac{2}{k} \to 0$$

as $k \to \infty$.

6. Show that a closed set in $\mathbb{R}^n$ is disconnected if and only if it is the union of two disjoint nonempty closed subsets.

"$\Rightarrow$" Suppose $S \subseteq \mathbb{R}^n$ is disconnected. Then there is a disconnection S_1, S_2 such that $S = S_1 \cup S_2$, $S_1 \cap \overline{S_2} = \emptyset$, $\overline{S_1} \cap S_2 = \emptyset$ and $S_1 \neq \emptyset$, $S_2 \neq \emptyset$.

Claim that S_1 and S_2 are closed. If S_1 were not closed, then there would be $x_n \in S_1$ such that $x_n \to x$ and $x \notin S_1$. Since $\overline{S_1} \cap S_2 = \emptyset$ then $x \notin S_2$. Therefore $x \notin S_1 \cup S_2 = S$.

On the other hand S closed and $x_n \in S_1 \subseteq S$ implies $x \in S$ which is a contradiction. Therefore S_1 is closed.
A similar argument shows S_2 is closed.

"$\Leftarrow$" Suppose $S = S_1 \cup S_2$ where $S_1 \cap S_2 = \emptyset$, $S_1 \neq \emptyset$, $S_2 \neq \emptyset$ and both are closed.
Since S_1 is closed then $S_1 = \overline{S_1}$. It follows $\overline{S_1} \cap S_2 = S_1 \cap S_2 = \emptyset$.
On the other hand S_2 closed implies $S_1 \cap \overline{S_2} = S_1 \cap S_2 = \emptyset$. This implies S_1, S_2 is a disconnection of S so S is disconnected.

8. Show that the closure of a connected set is connected.

Suppose S is connected but for contradiction $\overline{S}$ is disconnected.
Let T_1, T_2 be a disconnection of $\overline{S}$. Therefore

$$\overline{S} = T_1 \cup T_2, \quad T_1 \neq \emptyset, \; T_2 \neq \emptyset, \quad T_1 \cap \overline{T_2} = \emptyset \text{ and } \overline{T_1} \cap T_2 = \emptyset.$$

Define

$$S_1 = T_1 \cap S \quad \text{and} \quad S_2 = T_2 \cap S$$

Claim that S_1, S_2 is a disconnection of S. First note.

$$S = \bar{S} \cap S = (T_1 \cup T_2) \cap S = (T_1 \cap S) \cup (T_2 \cap S) = S_1 \cup S_2$$

Suppose $x \in \bar{S}_1$. Then there is $x_n \in S_1$ with $x_n \to x$.
Now $S_1 = T_1 \cap S$ implies $x_n \in T_1$ consequently $x \in \bar{T}_1$.
It follows that $\bar{S}_1 \subseteq \bar{T}_1$. Now

$$\bar{S}_1 \cap S_2 \subseteq \bar{T}_1 \cap S_2 = \bar{T}_1 \cap (T_2 \cap S) \subseteq \bar{T}_1 \cap T_2 = \emptyset$$

Similarly $\bar{S}_2 \in \bar{T}_1$ so

$$S_1 \cap \bar{S}_2 \subseteq S_1 \cap \bar{T}_2 = (T_1 \cap S) \cap \bar{T}_2 \subseteq T_1 \cap T_2 = \emptyset.$$

What's left is to show $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

By the previous problem we know T_1 is closed since $\bar{S}$ is closed. Thus, if $S_1 = \emptyset$ then $S = S_1$ implies $\bar{S} = \bar{S}_1 \subseteq \bar{T}_1$.
Therefore $\bar{S} \cap T_2 \subseteq \bar{T}_1 \cap T_2 = \emptyset$ and $T_2 \subseteq \bar{S}$ implies $T_2 = \emptyset$
which contradicts T_1, T_2 being a disconnection. Thus $S_1 \neq 0$

Similarly $S_2 \neq 0$.

This implies S_1, S_2 is a disconnection of S. However S is connected by hypothesis.

Therefore, the only alternative is $\bar{S}$ is also connected.
Thus the closure of a connected set is connected.