HW3 due Oct 4

Turn in (page 84) 2.6#10

Practice (page 41) 1.8#1, (page 84) 2.6#2, 2.6#11, (page 94-95) 2.7#5, 2.7#10 (page 100) 2.8#2

§1.8# **1.** A function $\mathbf{f} : S \to \mathbb{R}^m$ that satisfies

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|^\lambda \text{ for all } \mathbf{x}, \mathbf{y} \in S,$$

where C and λ are positive constants, is said to be **Hölder continuous on** S (with exponent λ). Show that if $\mathbf{f}$ is Hölder continuous on S, then $\mathbf{f}$ is uniformly continuous on S.

Given $\varepsilon > 0$ choose δ so that $C\delta^\lambda < \varepsilon$. Then $|x-y| < \delta$ implies $|f(x) - f(y)| \leq C|x-y|^\lambda < C\delta^\lambda < \varepsilon$.

Therefore, f is uniformly continuous.

§2.6# **2.** Calculate $\partial^2 u/\partial r \partial\theta$ if $u = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$. (See Example 4.)

Following Example 4 we have

$$\frac{\partial u}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} = (\cos\theta) f_x + (\sin\theta) f_y.$$

$$\frac{\partial u}{\partial \theta} = f_x \frac{\partial x}{\partial \theta} + f_y \frac{\partial y}{\partial \theta} = -(r\sin\theta) f_x + (r\cos\theta) f_y.$$

~~Setting~~ replacing u by $\frac{\partial u}{\partial r}$ in the second equation yields

$$\frac{\partial}{\partial\theta}\frac{\partial u}{\partial r} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right)\frac{\partial y}{\partial \theta} = -(r\sin\theta)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) + (r\cos\theta)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right)$$

Now

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial x}\left\{(\cos\theta) f_x + (\sin\theta) f_y\right\}$$

$$= \frac{\partial \cos\theta}{\partial x} f_x + \cos\theta\, f_{xx} + \frac{\partial \sin\theta}{\partial x} f_y + \sin\theta\, f_{xy}$$

Now $x = r\cos\theta$ implies $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ so that

$$\frac{\partial \cos\theta}{\partial x} = \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{r^3}$$

Similarly

$$\frac{\partial \sin\theta}{\partial x} = \frac{\partial}{\partial x}\frac{y}{\sqrt{x^2+y^2}} = \frac{xy}{(x^2+y^2)^{3/2}} = \frac{-xy}{r^3}$$

It follows that

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) = \frac{y^2}{r^3} f_x + \cos\theta\, f_{xx} - \frac{xy}{r^3} f_y + \sin\theta\, f_{xy}$$

$$= \frac{\sin^2\theta}{r} f_x + \cos\theta\, f_{xx} - \frac{\cos\theta\sin\theta}{r} f_y + \sin\theta\, f_{xy}.$$

To finish we next compute

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial y}\left\{(\cos\theta) f_x + (\sin\theta) f_y\right\}$$

$$= \frac{\partial \cos\theta}{\partial y} f_x + \cos\theta\, f_{xy} + \frac{\partial \sin\theta}{\partial y} f_y + \sin\theta\, f_{yy}.$$

Again $x = r\cos\theta$ implies $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ so that

$$\frac{\partial \cos\theta}{\partial y} = \frac{\partial}{\partial y}\frac{x}{\sqrt{x^2+y^2}} = \frac{3xy}{(x^2+y^2)^{3/2}} = -\frac{xy}{r^3}$$

Similarly

$$\frac{\partial \sin\theta}{\partial y} = \frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2} - y\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2+y^2-y^2}{(x^2+y^2)^{3/2}} = \frac{x^2}{r^3}.$$

Consequently

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right) = \frac{-xy}{r^3}f_x + \cos\theta\, f_{xy} + \frac{x^2}{r^3}f_y + \sin\theta\, f_{yy}$$

$$= \frac{-\cos\theta\sin\theta}{r}f_x + \cos\theta\, f_{xy} + \frac{\cos^2\theta}{r}f_y + \sin\theta\, f_{yy}$$

Putting everything together then yields

$$\frac{\partial}{\partial\theta}\frac{\partial u}{\partial r} = -(r\sin\theta)\left\{\frac{\sin^2\theta}{r}f_x + \cos\theta\, f_{xx} - \frac{\cos\theta\sin\theta}{r}f_y + \sin\theta\, f_{xy}\right\}$$

$$+ r\cos\theta\left\{-\frac{\cos\theta\sin\theta}{r}f_x + \cos\theta\, f_{xy} + \frac{\cos^2\theta}{r}f_y + \sin\theta\, f_{yy}\right\}.$$

$$= \left(-\sin^3\theta - \cos^2\theta\sin\theta\right)f_x + \left(\cos\theta\sin^2\theta + \cos^3\theta\right)f_y$$

$$-r\cos\theta\sin\theta\, f_{xx} + \left(-r\sin^2\theta + r\cos^2\theta\right)f_{xy} + r\cos\theta\sin\theta\, f_{yy}.$$

Simplifying as

$$-\sin^3\theta - \cos^2\sin\theta = -(\sin^2\theta + \cos^2\theta)\sin\theta = -\sin\theta$$

$$\cos\sin^2\theta + \cos^3\theta = \cos\theta(\sin^2\theta + \cos^2\theta) = \cos\theta$$

yields that

$$\frac{\partial}{\partial\theta}\frac{\partial u}{\partial r} = -\sin\theta\, f_x + \cos\theta\, f_y + r\cos\theta\sin\theta\,(f_{yy} - f_{xx}) + r\,(\cos^2\theta - \sin^2\theta)\, f_{xy}$$

§2.6 # **10.** Derive the following version of the product rule for partial derivatives: $\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha}(\alpha!/\beta!\gamma!)\partial^\beta f\partial^\gamma g$.

We begin by considering the product rule for regular derivatives.

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$(fg)''(x) = (f'g + fg')'(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

$$(fg)'''(x) = (f''g + 2f'g' + fg'')'(x)$$

$$= f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + g'''(x)$$

The pattern is

$$(fg)^{(k)}(x) = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} f^{(k-j)}(x)\, g^{(j)}(x).$$

We prove this by induction on k. The base case has already been shown. Now suppose the formula holds for k. Then

$$(fg)^{(k+1)}(x) = \frac{d}{dx}(fg)^{k}(x) = \frac{d}{dx}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!} f^{(k-j)}(x)\, g^{(j)}(x)$$

$$= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \left(f^{(k-j+1)}(x) g^{(j)}(x) + f^{(k-j)}(x) g^{(j+1)}(x) \right)$$

$$= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} f^{(k-j+1)}(x) g^{(j)}(x) + \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} f^{(k-j)}(x) g^{(j+1)}(x)$$

$$= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} f^{(k-j+1)}(x) g^{(j)}(x) + \sum_{j=1}^{k+1} \frac{k!}{(j-1)!(k-j+1)!} f^{(k-j+1)}(x) g^{(j)}(x)$$

$$= f^{(k+1)}(x) g(x) + \sum_{j=1}^{k} \left(\frac{k!}{j!(k-j)!} + \frac{k!}{(j-1)!(k-j+1)!} \right) f^{(k-j+1)}(x) g^{(j)}(x) + f(x) g^{(k+1)}(x)$$

Since

$$\frac{k!}{j!(k-j)!} + \frac{k!}{(j-1)!(k-j+1)!} = \frac{k!(k-j+1) + k!\,j}{j!(k-j+1)!} = \frac{(k+1)!}{j!(k+1-j)!}$$

it follows that

$$(fg)^{(k+1)}(x) = f^{(k+1)}(x) g(x) + \sum_{j=1}^{k} \frac{(k+1)!}{j!(k+1-j)!} f^{(k-j+1)}(x) g^{(j)}(x) + f(x) g^{(k+1)}(x)$$

$$= \sum_{j=0}^{k+1} \frac{(k+1)!}{j!(k+1-j)!} f^{(k-j+1)}(x) g^{(j)}(x)$$

which completes the induction step.

We now consider multi-indexes α of length n and perform induction on n obtain the desired result.

The base case when $n=1$ has already been shown.

For induction, suppose the formula

$$\partial^\alpha (fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!} \partial^\beta f\, \partial^\gamma g$$

holds for all multi-indexes of length n.

Let α be a multi-index of length $n+1$. Then

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \quad \text{where} \quad \alpha_i \in \mathbb{N}\cup\{0\}.$$

Let $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Then α' is a multi-index of length n and so by the induction hypothesis.

$$\partial^{\alpha'} (fg) = \sum_{\beta'+\gamma'=\alpha'} \frac{\alpha'!}{\beta'!\,\gamma'!} \partial^{\beta'} f\, \partial^{\gamma'} g$$

By definition

$$\partial^\alpha (fg) = \partial_{n+1}^{\alpha_{n+1}} \partial^{\alpha'} (fg) = \partial_{n+1}^{\alpha_{n+1}} \sum_{\beta'+\gamma'=\alpha'} \frac{\alpha'!}{\beta'!\,\gamma'!} \partial^{\beta'} f\, \partial^{\gamma'} g$$

$$= \sum_{\beta'+\gamma'=\alpha'} \frac{\alpha'!}{\beta'!\,\gamma'!} \partial_{n+1}^{\alpha_{n+1}} \left(\partial^{\beta'} f\, \partial^{\gamma'} g\right)$$

From the single variable case we have

$$\partial_{n+1}^{\alpha_{n+1}} \left(\partial^{\beta'} f\, \partial^{\gamma'} g\right) = \sum_{j=0}^{\alpha_{n+1}} \frac{\alpha_{n+1}!}{j!\,(\alpha_{n+1}-j)!} \left(\partial_{n+1}^{\alpha_{n+1}-j} \partial^{\beta'} f\right)\left(\partial_{n+1}^{j} \partial^{\gamma_1} g\right)$$

Substitution obtains

$$\partial^\alpha(fg) = \sum_{\beta'+\gamma'=\alpha'} \frac{\alpha'!}{\beta'!\gamma'!} \sum_{j=0}^{\alpha_{n+1}} \frac{\alpha_{n+1}!}{j!(\alpha_{n+1}-j)!} (\partial_{n+1}^{\alpha_{n+1}-j}\partial^{\beta'} f)(\partial_{n+1}^{j}\partial^{\gamma'} g)$$

Now, writing $\beta_{n+1}=\alpha_{n+1}-j$ and $\gamma_{n+1}=j$ yields

$$\partial^\alpha(fg) = \sum_{\beta'+\gamma'=\alpha'} \frac{\alpha'!}{\beta'!\gamma'!} \sum_{\beta_{n+1}+\gamma_{n+1}=\gamma_{n+1}} \frac{\alpha_{n+1}!}{\beta_{n+1}!\gamma_{n+1}!} (\partial_{n+1}^{\beta_{n+1}}\partial^{\beta'} f)(\partial_{n+1}^{\gamma_{n+1}}\partial^{\gamma'} g)$$

Since $\partial_{n+1}^{\beta_{n+1}}\partial^{\beta'} f = \partial^\beta f$ and $\partial_{n+1}^{\gamma_{n+1}}\partial^{\gamma'} g = \partial^\gamma g$ as well as $\alpha! = \alpha'!\alpha_{n+1}!$, $\beta! = \beta'!\beta_{n+1}!$ and $\gamma! = \gamma'!\gamma_{n+1}!$, we obtain that

$$\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial^\beta f \partial^\gamma g$$

which completes the induction.

§2.6 # **11.** Prove the following n-dimensional binomial theorem: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have $(\mathbf{x}+\mathbf{y})^\alpha = \sum_{\beta+\gamma=\alpha}(\alpha!/\beta!\gamma!)\mathbf{x}^\beta \mathbf{y}^\gamma$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a multi-index

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The one dimensional binomial theorem as given by Theorem 2 in the text states

$$(x_1+y_1)^{\alpha_1} = \sum_{\beta_1+\gamma_1=\alpha_1} \frac{\alpha_1!}{\beta_1!\,\gamma_1!} x_1^{\beta_1} y_1^{\gamma_1}$$

This evidently holds for any subscript as

$$(x_j+y_j)^{\alpha_j} = \sum_{\beta_j+\gamma_j=\alpha_j} \frac{\alpha_j!}{\beta_j!\,\gamma_j!} x_j^{\beta_j} y_j^{\gamma_j},$$

Substituting

$$(x+y)^\alpha = (x_1+y_1)^{\alpha_1}(x_2+y_2)^{\alpha_2}\cdots(x_n+y_n)^{\alpha_n}$$

$$= \left(\sum_{\beta_1+\gamma_1=\alpha_1} \frac{\alpha_1!}{\beta_1!\,\gamma_1!} x_1^{\beta_1} y_1^{\gamma_1}\right)\left(\sum_{\beta_2+\gamma_2=\alpha_2} \frac{\alpha_2!}{\beta_2!\,\gamma_2!} x_2^{\beta} y_2^{\gamma}\right)\cdots\left(\sum_{\beta_n+\gamma_n=\alpha_n} \frac{\alpha_n!}{\beta_n!\,\gamma_n!} x_n^{\beta_n} y_n^{\gamma_n}\right)$$

$$= \sum_{\beta_1+\gamma_1=\alpha_1}\sum_{\beta_2+\gamma_2=\alpha_2}\cdots\sum_{\beta_n+\gamma_n=\alpha_n} \frac{\alpha_1!}{\beta_1!\,\gamma_1!} x_1^{\beta_1} y_1^{\gamma_1} \frac{\alpha_2!}{\beta_2!\,\gamma_2!} x_2^{\beta_2} y_2^{\gamma_2} \cdots \frac{\alpha_n!}{\beta_n!\,\gamma_n!} x_n^{\beta_n} y_n^{\gamma_n}$$

$$= \sum_{\beta+\gamma=\alpha}\left(\frac{\alpha_1!}{\beta_1!\,\gamma_1!}\frac{\alpha_2!}{\beta_2!\,\gamma_2!}\cdots\frac{\alpha_n!}{\beta_n!\,\gamma_n!}\right)\left(x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n}\right)\left(y_1^{\gamma_1}y_2^{\gamma_2}\cdots y_n^{\gamma_n}\right)$$

Using $x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} = x^{\beta}$ and $y_1^{\gamma_1} y_2^{\gamma_2} \cdots y_n^{\gamma_n} = y^{\gamma}$ as well as the definitions for $\alpha!$, $\beta!$ and $\gamma!$ it follows that

$$(x+y)^{\alpha} = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!} x^{\beta} y^{\beta}$$

which was to be shown.

§2.7# **5.** Find the Taylor polynomial of order 4 based at $\mathbf{a} = (0,0)$ for each of the following functions. Don't compute any derivatives; use Proposition 2.65.

a. $f(x,y) = x\sin(x+y)$.
b. $e^{xy}\cos(x^2+y^2)$.
c. $e^{x-2y}/(1+x^2-y)$.

First recall

2.65 Proposition. *The Taylor polynomials of degree k about $a = 0$ of the functions*

$$e^x, \qquad \cos x, \qquad \sin x, \qquad (1-x)^{-1}$$

are, respectively,

$$\sum_{0\le j\le k} \frac{x^j}{j!}, \qquad \sum_{0\le j\le k/2} \frac{(-1)^j x^{2j}}{(2j)!}, \qquad \sum_{0\le j\le (k-1)/2} \frac{(-1)^j x^{2j+1}}{(2j+1)!}, \qquad \sum_{0\le j\le k} x^j.$$

(a) Note $\sin(x+y) \approx \displaystyle\sum_{0\le j\le (k-1)/2} \frac{(-1)^j (x+y)^{2j+1}}{(2j+1)!}$

and $(x+y)^{2j+1} = \displaystyle\sum_{l=0}^{2j+1} \binom{2j+1}{l} x^{2j+1-l} y^{l}$

Substituting yields

$$x\sin(x+y) \sim x \sum_{0\le j\le (k-1)/2} \frac{(-1)^j}{(2j+1)!} \sum_{l=0}^{2j+1} \binom{2j+1}{l} x^{2j+1-l} y^l$$

$$\sim \sum_{0\le j\le (k-1)/2} \frac{(-1)^j}{(2j+1)!} \sum_{l=0}^{2j+1} \binom{2j+1}{l} x^{2j+2-l} y^l$$

Since $2\left(\frac{k-1}{2}\right)+2 = k-1+2 = k+1$ we stop the first sum at $(k-2)/2$ to obtain the Taylor polynomial T_k of degree k

$$T_k(x,y) = \sum_{0\le j\le (k-2)/2} \frac{(-1)^j}{(2j+1)!} \sum_{l=0}^{2j+1} \binom{2j+1}{l} x^{2j+2-l} y^l$$

In particular

$$T_4(x,y) = \underbrace{x^2 + xy}_{j=0} \underbrace{- \frac{1}{6}\left(\binom{3}{0}x^4 + \binom{3}{1}x^3y + \binom{3}{2}x^2y^2 + \binom{3}{3}xy^3\right)}_{j=1}$$

$$= x^2 + xy - \frac{1}{6}\left(x^4 + 3x^3y + 3x^2y^2 + xy^3\right).$$

(b) $$e^{xy} \sim 1 + xy + \frac{1}{2}x^2y^2$$

$$\cos(x^2+y^2) \sim 1 - \frac{1}{2}(x^2+y^2)^2 \sim 1 - \frac{1}{2}\left(x^4 + 2x^2y^2 + y^4\right)$$

Therefore

$$e^{xy}\cos(x^2+y^2) \sim \left(1+xy+\tfrac{1}{2}x^2y^2\right)\left(1-\tfrac{1}{2}x^4-x^2y^2-\tfrac{1}{2}y^4\right)$$

$$\sim 1+xy+\tfrac{1}{2}x^2y^2-\tfrac{1}{2}x^4-x^2y^2-\tfrac{1}{2}y^4$$

$$\sim 1+xy-\tfrac{1}{2}\left(x^4+x^2y^2+y^4\right)$$

and so

$$T_4(x,y) = 1+xy-\tfrac{1}{2}\left(x^4+x^2y^2+y^4\right)$$

(c) $$e^{x-2y} \sim 1+(x-2y)+\tfrac{1}{2}(x-2y)^2+\tfrac{1}{6}(x-2y)^3+\tfrac{1}{24}(x-2y)^4$$

$$\frac{1}{1+x^2-y} = \frac{1}{1-(y-x^2)} \sim 1+(y-x^2)+(y-x^2)^2+(y-x^2)^3+(y-x^2)^4$$

$$\sim 1+y-x^2+y^2-2x^2y+x^4+y^3-3y^2x^2+y^4$$

Therefore

$$e^{x-2y}\frac{1}{1+x^2-y} \sim \left(1+(x-2y)+\tfrac{1}{2}(x-2y)^2+\tfrac{1}{6}(x-2y)^3+\tfrac{1}{24}(x-2y)^4\right)\left(1+y-x^2+y^2-2x^2y+x^4+y^3-3y^2x^2+y^4\right)$$

So

$$T_4(x,y) = 1+(x-2y)+\tfrac{1}{2}(x-2y)^2+\tfrac{1}{6}(x-2y)^3+\tfrac{1}{24}(x-2y)^4+y\left(1+(x-2y)+\tfrac{1}{2}(x-2y)^2+\tfrac{1}{6}(x-2y)^3\right)$$
$$+(y^2-x^2)\left(1+(x-2y)+\tfrac{1}{2}(x-2y)^2\right)+(y^3-2x^2y)\left(1+(x-2y)\right)+x^4-3y^2x^2+y^4.$$

Upon simplification, assuming no mistake,

$$T_4(x,y) = 1 + x - y - \frac{1}{2}x^2 - xy + y^2 - \frac{5}{6}x^3 - \frac{1}{2}x^2y + xy^2 - \frac{1}{3}y^3$$
$$+ \frac{13}{24}x^4 - \frac{1}{6}x^3y - \frac{1}{2}x^2y^2 - \frac{1}{3}xy^3 + \frac{1}{3}y^4$$

§2.7 # **10.** Suppose f is of class C^k on an open convex set $S \subset \mathbb{R}^n$ and its kth-order derivatives, $\partial^\alpha f$ with $|\alpha| = k$, satisfy

$$|\partial^\alpha f(\mathbf{y}) - \partial^\alpha f(\mathbf{x})| \le C|\mathbf{y} - \mathbf{x}|^\lambda \qquad (\mathbf{x}, \mathbf{y} \in S),$$

where C and λ are positive constants (cf. Exercise 1 in §1.8). Use (2.70) to show that there is another positive constant C' such that

$$|R_{\mathbf{a},k}(\mathbf{h})| \le C'|\mathbf{h}|^{k+\lambda} \qquad (\mathbf{a} \in S \text{ and } \mathbf{a} + \mathbf{h} \in S).$$

Recall

$$\text{(2.70)} \qquad R_{\mathbf{a},k}(\mathbf{h}) = k \sum_{|\alpha|=k} \frac{\mathbf{h}^\alpha}{\alpha!} \int_0^1 (1-t)^{k-1} \big[\partial^\alpha f(\mathbf{a} + t\mathbf{h}) - \partial^\alpha f(\mathbf{a})\big]\, dt.$$

Therefore

$$|R_{a,k}(h)| \le k \sum_{|\alpha|=k} \frac{|h^\alpha|}{\alpha!} \int_0^1 (1-t)^{k-1} \, C\,|a + th - a|^\lambda \, dt$$

$$\le k \sum_{|\alpha|=k} \frac{|h|^k}{\alpha!} C \int_0^1 (1-t)^{k-1} t^\lambda |h|^\lambda \, dt$$

$$= k \sum_{|\alpha|=k} \frac{|h|^{k+\lambda}}{\alpha!} C \int_0^1 (1-t)^{k-1} t^\lambda \, dt$$

Since $|1-t| \leq 1$ and $|t|^{\lambda} \leq 1$ when $t \in (0,1)$ then

$$\int_0^1 (1-t)^{k-1} t^{\lambda} \, dt \leq \int_0^1 dt = 1$$

Therefore

$$|R_{a,k}(h)| \leq C' |h|^{k+\lambda}$$

where $C' = k C \sum_{|\alpha|=k} \frac{1}{\alpha!}$

By the multinomial theorem

$$n^k = (\underbrace{1+1+\cdots+1}_{n \text{ terms}})^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!}(1,1,\dots,1)^{\alpha} = k! \sum_{|\alpha|=k} \frac{1}{\alpha!}$$

Therefore

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{n^k}{k!}$$

It follows that C' may be simplified as

$$C' = kC \frac{n^k}{k!} = C \frac{n^k}{(k-1)!} .$$

§2.8 # **2.** What are the conditions on a, b, c for $f(x, y) = ax^2 + bxy + cy^2$ to have a minimum, maximum, or saddle point at the origin?

Note first that $\nabla f(x,y) = (2ax + by,\ bx + 2cy)$.

Therefore $\nabla f(0,0) = (0,0)$ shows the origin is a critical point.

$$H(x,y) = \begin{pmatrix} \partial_1^2 f(x,y) & \partial_1\partial_2 f(x,y) \\ \partial_1\partial_2 f(x,y) & \partial_2^2 f(x,y) \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

Recall

2.82 Theorem. *Suppose f is of class C^2 on an open set in $\mathbb{R}^2$ containing the point $\mathbf{a}$, and suppose $\nabla f(\mathbf{a}) = \mathbf{0}$. Let $\alpha = \partial_1^2 f(\mathbf{a})$, $\beta = \partial_1\partial_2 f(\mathbf{a})$, $\gamma = \partial_2^2 f(\mathbf{a})$. Then:*

a. *If $\alpha\gamma - \beta^2 < 0$, f has a saddle point at $\mathbf{a}$.*
b. *If $\alpha\gamma - \beta^2 > 0$ and $\alpha > 0$, f has a local minimum at $\mathbf{a}$.*
c. *If $\alpha\gamma - \beta^2 > 0$ and $\alpha < 0$, f has a local maximum at $\mathbf{a}$.*
d. *If $\alpha\gamma - \beta^2 = 0$, no conclusion can be drawn.*

Setting $\alpha = 2a$, $\beta = b$ and $\gamma = 2c$ yields $\alpha\gamma - \beta^2 = 4ac - b^2$. Thus we have that

$4ac - b^2 < 0$ implies f has a saddle point at $(0,0)$.

$4ac - b^2 > 0$ and $a > 0$ implies f has a local minimum at $(0,0)$

$4ac - b^2 > 0$ and $a < 0$ implies f has a local minimum at $(0,0)$

Since $a = 0$ implies $4ac - b^2 = -b^2 < 0$ is a saddle, it remains to consider the case where $4ac - b^2 = 0$.

Now, if $b^2 = 4ac$ then $ac > 0$.

Case $a > 0$ and $c > 0$.

$$f(x,y) = ax^2 + bxy + cy^2 = ax^2 \pm 2\sqrt{ac}\,xy + cy^2 = (\sqrt{a}x \pm \sqrt{c}y)^2$$

This shows that $(0,0)$ is a minimum.

Case $a < 0$ and $c < 0$

$$f(x,y) = ax^2 + bxy + cy^2 = -(-ax^2 - bxy - cy^2)$$
$$= -(-ax^2 \pm 2\sqrt{(-a)(-c)}\,xy + -cy^2)$$
$$= -(\sqrt{-a}x \pm \sqrt{-c}y)^2$$

In this case $(0,0)$ is a maximum.