HW4 due Nov 1

Turn in (page 133) 3.3#5ab

Practice (page 119) 3.1#2, (page 125) 3.2#4, (page 139) 3.4#4abcde

§3.1# **2.** Show that the equation $x^2 + 2xy + 3y^2 = c$ can be solved either for y as a C^1 function of x or for x as a C^1 function of y (but perhaps not both) near any point (a, b) such that $a^2 + 2ab + 3b^2 = c$, provided that $c > 0$. What happens if $c = 0$ or if $c < 0$?

One could solve for x or y using the quadratic formula, but the question is probably about using the implicit function theorem.

3.1 Theorem (The Implicit Function Theorem for a Single Equation). *Let $F(\mathbf{x}, y)$ be a function of class C^1 on some neighborhood of a point $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$. Suppose that $F(\mathbf{a}, b) = 0$ and $\partial_y F(\mathbf{a}, b) \neq 0$. Then there exist positive numbers r_0, r_1 such that the following conclusions are valid.*

a. *For each $\mathbf{x}$ in the ball $|\mathbf{x} - \mathbf{a}| < r_0$ there is a unique y such that $|y - b| < r_1$ and $F(\mathbf{x}, y) = 0$. We denote this y by $f(\mathbf{x})$; in particular, $f(\mathbf{a}) = b$.*

b. *The function f thus defined for $|\mathbf{x} - \mathbf{a}| < r_0$ is of class C^1, and its partial derivatives are given by*

$$\partial_j f(\mathbf{x}) = -\frac{\partial_j F(\mathbf{x}, f(\mathbf{x}))}{\partial_y F(\mathbf{x}, f(\mathbf{x}))}. \tag{3.2}$$

Define $F(x,y) = x^2 + 2xy + 3y^2 - c$. Clearly $F \in C^1(\mathbb{R}^2)$. Now suppose $F(a,b) = 0$. Compute

$$\partial_y F(x,y) = 2x + 6y \quad \text{and} \quad \partial_x F(x,y) = 2x + 2y.$$

Since $a^2 + 2ab + 3b^2 - c = 0$ and $c > 0$ it follows that both a and b can't be identically zero.

If $\partial_y F(a,b) \neq 0$ then the hypothesis of the theorem are satisfied and there is a C' function $f(x)$ such that for (x,y) in a neighborhood of (a,b) that

$$F(x,y)=0 \quad \text{if and only if} \quad y=f(x).$$

If $\partial_y F(a,b)=0$ then claim that $\partial_x F(a,b)\neq 0$. Since $\partial_y F(a,b) = 2a+6b=0$ it follows $a=-3b$. Therefore if not both $a=0$ and $b=0$ then both $a\neq 0$ and $b\neq 0$. Now

$$\partial_x F(a,b) = 2a+2b = -6b+2b = -4b \neq 0$$

Now apply the theorem with the roles of x and y switched. It follows there is a C' function $g(y)$ such that for (x,y) in a neighborhood of (a,b) that

$$F(x,y)=0 \quad \text{if and only if} \quad x=g(y).$$

Consider the case $c=0$. Now $F(a,b) = a^2+2ab+3b^2=0$ does not imply $(a,b)\neq 0$. Consequently, for $(a,b)=0$ we have

$$\partial_y F(0,0)=0 \quad \text{and} \quad \partial_x F(0,0)=0$$

Further $F(x,y) = x^2+2xy+3y^2 = (x+y)^2+2y^2=0$ implies that $(x,y)=0$ is the only solution. A single point can't be written as the graph of a C' function.

If $c<0$ then

$$F(x,y) = x^2+2xy+3y^2-c = x^2+2xy+3y^2+|c|$$

$$= (x+y)^2+y^2+|c|>0$$

and so there are no points (a,b) such that $F(a,b)=0$ and the hypothesis of the implicit function theorem are vacuous.

§3.2 # **4.** Let $\varphi(s) = s^2$ if $s \geq 0$, $\varphi(s) = -s^2$ if $s < 0$.

a. Show that φ is of class C^1, even at $s = 0$.

b. Let $\mathbf{f}(t) = (\varphi(\cos t), \varphi(\sin t))$. Show that $\{\mathbf{f}(t) : t \in \mathbb{R}\}$ is the square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. For which values of t is $\mathbf{f}'(t) = \mathbf{0}$? What are the corresponding points $\mathbf{f}(t)$?

(a) Clearly φ is differentiable by the power rule when $s \neq 0$. When $s=0$ consider the left and right limits

$$\lim_{h\to 0^+} \frac{\varphi(h)-\varphi(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0$$

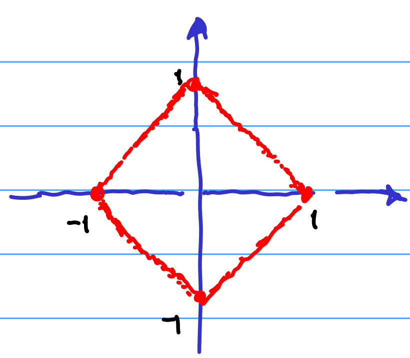
and

$$\lim_{h\to 0^-} \frac{\varphi(h)-\varphi(0)}{h} = \lim_{h\to 0^+} \frac{-h^3-0}{h} = \lim_{h\to 0^+} (-h) = 0,$$

Since both limits are the same, it follows that

$$\lim_{h\to 0} \frac{\varphi(h)-\varphi(0)}{h} = 0$$ and so $\varphi'(0)$ exists and $\varphi'(0)=0$.

(b) Let $f(t) = (\varphi(\cos t), \varphi(\sin t))$. The square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$ is S



Since $f(t)$ is 2π periodic it is enough to consider $t \in [0, 2\pi]$. Further break this interval down into four pieces

$$[0, 2\pi] = \left[0, \tfrac{\pi}{2}\right] \cup \left[\tfrac{\pi}{2}, \pi\right] \cup \left[\pi, \tfrac{3\pi}{2}\right] \cup \left[\tfrac{3\pi}{2}, 2\pi\right]$$

Case $t \in [0, \frac{\pi}{2}]$ then $\sin t \geq 0$ and $\cos t \geq 0$. It follows that

$$(\varphi(\cos t), \varphi(\sin t)) = (\cos^2 t, \sin^2 t)$$

The Pythagorean theorem implies $\cos^2 t + \sin^2 t = 1$ therefore the points fall on the line $x + y = 1$. Since $\cos^2 t \in [0,1]$ and similarly for every $x \in [0,1]$ there is $t \in [0, \pi/2]$ such that $x = \cos^2 t$ the map $t \to (\varphi(\cos t), \varphi(\sin t))$ corresponds one-to-one as $t \in [0, \pi/2]$ to the line segment connecting $(1,0)$ to $(0,1)$.

Case $t \in [\pi/2, \pi]$ then $\sin t \geq 0$ and $\cos t \leq 0$. It follows that

$$(\varphi(\cos t), \varphi(\sin t)) = (-\cos^2 t, \sin^2 t)$$

The Pythagorean theorem implies $\cos^2 t + \sin^2 t = 1$ therefore the points fall on the line $-x + y = 1$. Since $-\cos^2 t \in [-1,0]$ and similarly for every $x \in [-1,0]$ there is $t \in [\pi/2, \pi]$ such that $x = -\cos^2 t$ the map $t \to (\varphi(\cos t), \varphi(\sin t))$ corresponds one-to-one as $t \in [\pi/2, \pi]$ to the line segment connecting $(0,1)$ to $(-1,0)$.

Case $t \in [\pi, 3\pi/2]$ then $\sin t \leq 0$ and $\cos t \leq 0$. It follows that

$$(\varphi(\cos t), \varphi(\sin t)) = (-\cos^2 t, -\sin^2 t)$$

The Pythagorean theorem implies $\cos^2 t + \sin^2 t = 1$ therefore the points fall on the line $-x - y = 1$. Since $-\cos^2 t \in [-1,0]$ and similarly for every $x \in [-1,0]$ there is $t \in [\pi, 3\pi/2]$ such that $x = -\cos^2 t$ the map $t \to (\varphi(\cos t), \varphi(\sin t))$ corresponds one-to-one as $t \in [\pi, 3\pi/2]$ to the line segment connecting $(-1,0)$ to $(0,-1)$.

Case $t \in [\frac{3\pi}{2}, 2\pi]$ then $\sin t \le 0$ and $\cos t \ge 0$. It follows that

$$(\varphi(\cos t), \varphi(\sin t)) = (\cos^2 t, -\sin^2 t)$$

The Pythagorean theorem implies $\cos^2 t + \sin^2 t = 1$ therefore the points fall on the line $x - y = 1$. Since $\cos^2 t \in [0,1]$ and similarly for every $x \in [0,1]$ there is $t \in [\frac{3\pi}{2}, 2\pi]$ such that $x = \cos^2 t$ the map $t \to (\varphi(\cos t), \varphi(\sin t))$ corresponds one-to-one as $t \in [\frac{3\pi}{2}, 2\pi]$ to the line segment connecting $(0,-1)$ to $(1,0)$.

§3.3 #5. Let S be the circle formed by intersecting the plane $x + z = 1$ with the sphere $x^2 + y^2 + z^2 = 1$.

a. Find a parametrization of S.

b. Find parametric equations for the tangent line to S at the point $(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2})$.

(a) $x^2 + y^2 + z^2 = 1$, $\quad z = 1 - x$, $\quad x^2 + y^2 + (1-x)^2 = 1$

$x^2 + y^2 + x^2 - 2x + 1 = 1$, $\quad 2x^2 - 2x + y^2 = 0$

$2(x^2 - x) + y^2 = 0$, $\quad 2(x - \frac{1}{2})^2 - \frac{1}{2} + y^2 = 0$

$2(x - \frac{1}{2})^2 + y^2 = \frac{1}{2}$, $\quad 4(x - \frac{1}{2})^2 + 2y^2 = 1$

Let $\cos\theta = 2(x - \frac{1}{2})$ and $\sin\theta = \sqrt{2}\, y$. Then $\cos^2\theta + \sin^2\theta = 1$ is equivalent to the original curve.

Solving back for x and y gives

$\frac{1}{2}\cos\theta = x - \frac{1}{2}$ $\quad$ so $\quad$ $x = \frac{1}{2} + \frac{1}{2}\cos\theta$

$\sin\theta = \sqrt{2}\, y$ $\quad$ so $\quad$ $y = \frac{1}{\sqrt{2}}\sin\theta$.

Since $z = 1-x$ then $z = \frac{1}{2} - \frac{1}{2}\cos\theta$

Therefore the parameterization is

$$(x,y,z) = \left(\frac{1}{2}\cos\theta + \frac{1}{2}, \frac{1}{\sqrt{2}}\sin\theta, \frac{1}{2} - \frac{1}{2}\cos\theta\right)$$

(b) Solve for θ so

$$\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \left(\frac{1}{2}\cos\theta + \frac{1}{2}, \frac{1}{\sqrt{2}}\sin\theta, \frac{1}{2} - \frac{1}{2}\cos\theta\right)$$

Now $\cos\theta = 0$ implies $\theta = \frac{\pi}{2} + k\pi$

$\sin\theta = -1$ implies $\theta = \frac{3\pi}{2} + 2\pi l$

Evidently $\theta = \frac{3\pi}{2}$ satisfies both equations with $k=1$ and $l=0$.

A tangent vector (not unit length) is

$$\frac{d}{d\theta}\left(\frac{1}{2}\cos\theta + \frac{1}{2}, \frac{1}{\sqrt{2}}\sin\theta, \frac{1}{2} - \frac{1}{2}\cos\theta\right)\Bigg|_{\theta = \frac{3\pi}{2}}$$

$$= \left(-\frac{1}{2}\sin\theta, \frac{1}{\sqrt{2}}\cos\theta, \frac{1}{2}\sin\theta\right)\Bigg|_{\theta=\frac{3\pi}{2}} = \left(\frac{1}{2}, 0, -\frac{1}{2}\right)$$

Therefore a parametric equation for the tangent is

$$l(t) = \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) + t\left(\frac{1}{2}, 0, -\frac{1}{2}\right) = \left(\frac{1+t}{2}, -\frac{1}{\sqrt{2}}, \frac{1-t}{2}\right)$$

4. Let $(u, v) = \mathbf{f}(x, y) = (x - y,\ xy)$.

a. Sketch some of the curves $x - y$ = constant and xy = constant in the xy-plane. Which regions in the xy-plane map onto the rectangle in the uv-plane given by $0 \le u \le 1$, $1 \le v \le 4$? There are two of them; draw a picture of them.

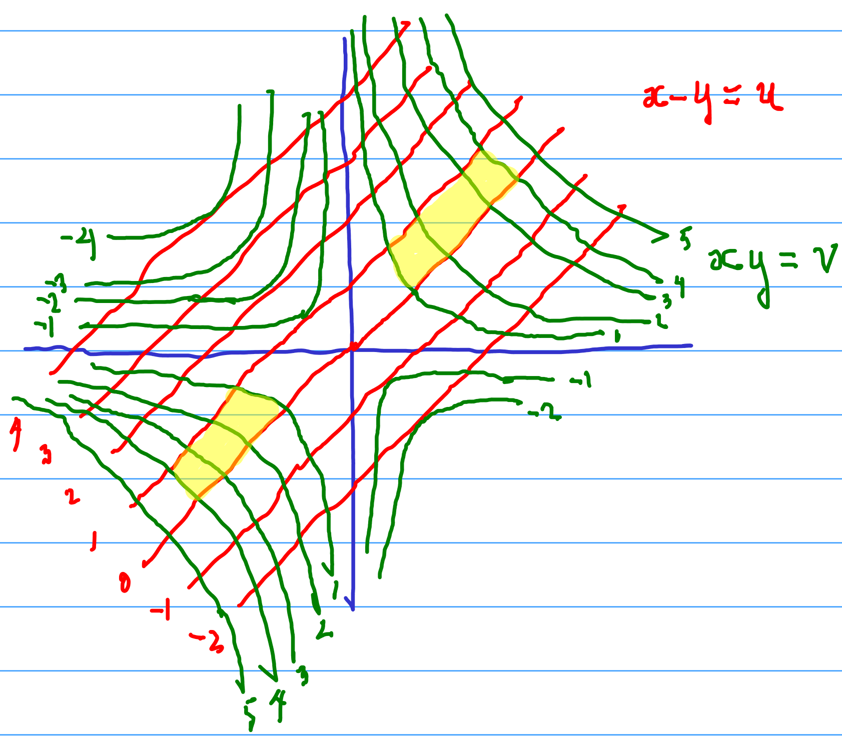
b. Compute the derivative $D\mathbf{f}$ and the Jacobian $J = \det D\mathbf{f}$.

c. The Jacobian J vanishes at (a, b) precisely when the gradients $\nabla u(a, b)$ and $\nabla v(a, b)$ are linearly dependent, i.e., when the level sets of u and v passing through a and b are tangent to each other. (If this doesn't seem obvious at first, think about it!) Use your sketch of the level sets in (a) to show pictorially that this assertion is correct.

d. Notice that $\mathbf{f}(2, -3) = (5, -6)$. Compute explicitly the local inverse $\mathbf{g}$ of $\mathbf{f}$ such that $\mathbf{g}(5, -6) = (2, -3)$ and also compute its derivative $D\mathbf{g}$.

e. Show by explicit calculation that the matrices $D\mathbf{f}(2, -3)$ and $D\mathbf{g}(5, -6)$ are inverses of each other.

(a)

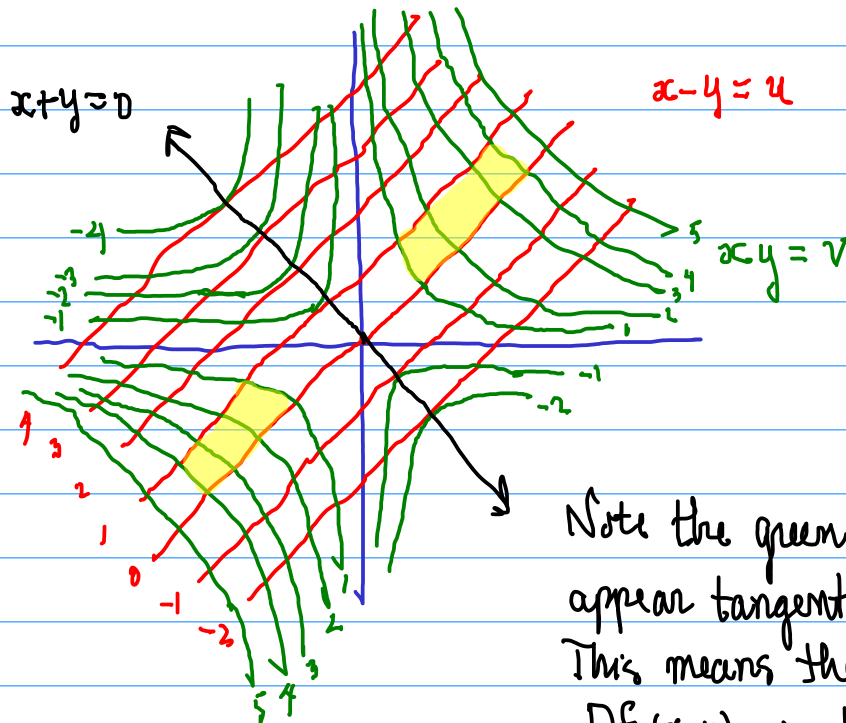


(b) $f(x,y) = (x-y,\ xy)$

$$Df(x,y) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ y & x \end{bmatrix}$$

$$J(x,y) = \det Df(x,y) = \det \begin{bmatrix} 1 & -1 \\ y & x \end{bmatrix} = x+y.$$

(c)



Note the green and red lines appear tangent along $x+y=0$. This means the columns in $Df(x,y)$ are dependent at those points.

(d) $f(x,y) = (x-y,\ xy)$

$f(2,-3) = (2-(-3),\ 2(-3)) = (5,-6)$

Now set $x-y=u$ and $xy=v$ and solve for x and y

$x = u+y$

substitute $(u+y)y = v$, $\quad y^2+uy-v=0$

By the quadratic formula

$$y = \frac{-u \pm \sqrt{u^2+4v}}{2}$$

Choose the sign so plugging in $(u,v)=(5,-6)$ yields $y=-3$.

$$\frac{-5\pm\sqrt{25-24}}{2}=\frac{-5\pm 1}{2}=-2,\ -3$$

Therefore
$$y=\frac{-u-\sqrt{u^2+4v}}{2}.$$

Now
$$x=u+y=\frac{u-\sqrt{u^2+4v}}{2}.$$

Check that $x=2$ when $(u,v)=(5,-6)$.

$$x=\frac{5-\sqrt{25-24}}{2}=\frac{5-1}{2}=2.$$

The inverse locally at $(x,y)=(2,-3)$

$$g(u,v)=\left(\frac{u-\sqrt{u^2+4v}}{2},\ \frac{-u-\sqrt{u^2+4v}}{2}\right).$$

Consequently

$$Dg(u,v)=\begin{bmatrix} \frac{1}{2}-\frac{u}{2\sqrt{u^2+4v}} & \frac{-1}{\sqrt{u^2+4v}} \\ -\frac{1}{2}-\frac{u}{2\sqrt{u^2+4v}} & \frac{-1}{\sqrt{u^2+4v}} \end{bmatrix}$$

(e) Recall

$$Df(x,y) = \begin{bmatrix} 1 & -1 \\ y & x \end{bmatrix} \quad \text{and} \quad Dg(u,v) = \begin{bmatrix} \frac{1}{2} - \frac{u}{2\sqrt{u^2+4v}} & \frac{-1}{\sqrt{u^2+4v}} \\ -\frac{1}{2} - \frac{u}{2\sqrt{u^2+4v}} & \frac{-1}{\sqrt{u^2+4v}} \end{bmatrix}$$

Therefore $\quad Df(2,-3) = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$

and

$$Dg(5,-6) = \begin{bmatrix} \frac{1}{2} - \frac{5}{2\sqrt{25-24}} & \frac{-1}{\sqrt{25-24}} \\ -\frac{1}{2} - \frac{5}{2\sqrt{25-24}} & \frac{-1}{\sqrt{25-24}} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix}$$

Now compute

$$\begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -2+3 & -1+1 \\ 6-6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore $Df(2,-3)$ and $Df(5,-6)$ are inverses of each other.