HW5 due Nov 15

Turn in (page 167) 4.2#7

Practice (page 158) 4.1#6, (page 167) 4.2#3,4,5 (page 176) 4.3#5abc

**6.** Let $\{x_k\}$ be a convergent sequence in $\mathbb{R}$. Show that the set $\{x_1, x_2, \ldots\}$ has zero content.

Since $x_k$ converges, it has a limit $\lim_{k\to\infty} x_k = x$.

Let $R_\varepsilon(x) = [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$ be the interval of length $\varepsilon$ centered at $x$.

Let $\varepsilon > 0$ and choose $K$ so large that $k \geq K$ implies that $|x - x_k| < \frac{\varepsilon}{2}$. Thus, $x_k \in R_{\varepsilon/2}(x)$ for all $k \geq K$.

Note that $x_{K-1} \in R_{\varepsilon/2^2}(x_{K-1})$, $x_{K-2} \in R_{\varepsilon/2^3}(x_{K-2})$ and in general that $x_j \in R_{\varepsilon/2^{K-j+1}}(x_j)$ for $j = 1, 2, \ldots, K-1$.

Now let $R_j = \begin{cases} R_{\varepsilon/2^{K-j+1}}(x_j) & \text{for } j = 1, 2, \ldots, K-1 \\ R_{\varepsilon/2}(x) & \text{for } j = K \end{cases}$

It follows that

$$\{x_k\} = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_{K-1}\} \cup \{x_K, x_{K+1}, \ldots\}$$
$$\subseteq R_1 \cup R_2 \cup \cdots \cup R_{K-1} \cup R_K = \bigcup_{j=1}^{K} R_j$$

and that

$$\sum_{j=1}^{K} |R_j| = \sum_{j=1}^{K-1} |R_{\varepsilon/2^{K-j+1}}(x_j)| + |R_{\varepsilon/2}(x)| = \sum_{j=1}^{K} \frac{\varepsilon}{2^{K-j+1}}$$
$$= \varepsilon\left(\frac{1}{2^K} + \frac{1}{2^{K-1}} + \cdots + \frac{1}{2}\right) = \varepsilon \, \frac{1/2 - 1/2^{K-1}}{1 - 1/2} < \varepsilon.$$

Therefore $\{x_k\}$ has zero content.

**3.** Let $S$ be a bounded set in $\mathbb{R}^2$. Show that $S$ and $S^{\text{int}}$ have the same inner area. (*Hint:* For any rectangle contained in $S$, there are slightly smaller rectangles contained in $S^{\text{int}}$.)

Let $S \subseteq [a,b]\times[c,d]$. By definition the inner area is

$$\underline{A} = \underline{I}_R(\chi_S) = \sup\{s_P(f) : P \text{ is a partition of } [a,b]\times[c,d]\}$$

Here $P = \{x_0, \dots, x_J; y_0, \dots, y_K\}$ where $x_0 = a < x_1 < \cdots < x_J = b$

and $y_0 = c < y_1 < \cdots < y_K = d.$

Also $R_{jk} = [x_{j-1}, x_j]\times[y_{k-1}, y_k]$ and $|R_{jk}| = (x_j - x_{j-1})(y_k - y_{k-1})$ so that

$$s_P(f) = \sum_{j,k} m_{jk}\,|R_{jk}| \quad \text{where} \quad m_{jk} = \inf\{f(x,y) : (x,y)\in R_{jk}\}.$$

Setting $f(x,y) = \chi_S(x,y)$ we obtain

$$m_{jk} = \begin{cases} 1 & \text{if } R_{jk} \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Given $\varepsilon>0$ choose $\delta>0$ so $2\delta < \min\{\Delta x_j : j=1,\dots,J\}\cup\{\Delta y_k : k=1,\dots,K\}$

and further so $\delta < \varepsilon / 2[K(b-a) + J(d-c)]$.

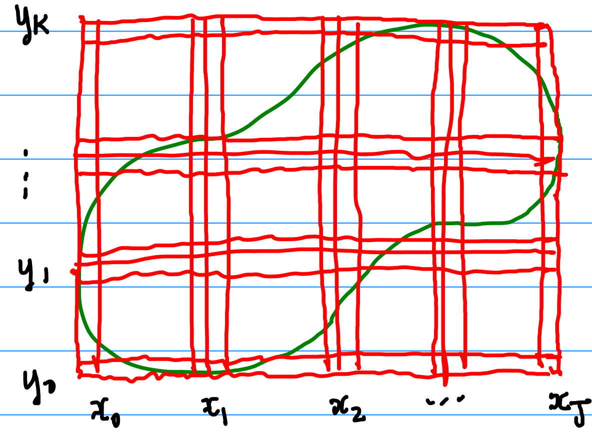
refine the partition $P$ to obtain $P_\delta$ as

$$x_0 < x_0+\delta < x_1-\delta < x_1 < x_1+\delta < \cdots < x_J-\delta < x_J$$

$$y_0 < y_0+\delta < y_1-\delta < y_1 < y_1+\delta < \cdots < y_K-\delta < y_K$$

Note that $2J$ points have been added to the $x$'s and $2K$ to the $y$'s

In pictures the refined partition $P_\delta$ looks like




Note that

$$[x_{j-1}+\delta,\ x_j-\delta]\times[y_{k-1}+\delta,\ y_k-\delta] \subseteq (x_{j-1},x_j)\times(y_{k-1},y_k) = R_{jk}^{int}$$

Therefore if $R_{jk}\subseteq S$

then $[x_{j-1}+\delta,\ x_j-\delta]\times[y_{k-1}+\delta,\ y_k-\delta] \subseteq S^{int}$.

Let $R_{jk}^{\delta} = [x_{j-1}+\delta,\ x_j-\delta]\times[y_{k-1}+\delta,\ y_k-\delta]$, Then

$$|R_{jk}| - |R_{jk}^{\delta}| = 2\delta\,\Delta x_j + 2\delta\,\Delta y_j - 4\delta^2 < 2\delta(\Delta x_j+\Delta y_j).$$

Let $m_{jk}^{int} = \inf\{\chi_{S^{int}} : (x,y)\in R_{jk}^{\delta}\}$ then $m_{jk}^{int} \geq m_{j,k}$.

Since $0\leq \chi_{S^{int}}\leq 1$ then

$$s_{P_\delta}(\chi_{S^{int}}) \geq \sum_{jk} m_{jk}^{int}\,|R_{jk}^{\delta}| \geq \sum_{j,k} m_{jk}\,|R_{jk}^{\delta}|$$

$$= \sum_{j,k} m_{jk}\,|R_{jk}| - \sum_{j,k} m_{jk}\left(|R_{jk}|-|R_{jk}^{\delta}|\right)$$

$$s_{P_\delta}(\chi_{S^{int}}) \geq s_P(\chi_S) - 2\delta \sum_{j,k} m_{jk} (\Delta x_j + \Delta y_k)$$

$$\geq s_P(\chi_S) - 2\delta \sum_{j,k} (\Delta x_j + \Delta y_k)$$

$$= s_P(\chi_S) - 2\delta \left[K(b-a) + J(d-c)\right] > s_P(\chi_S) - \varepsilon$$

In summary for every $P$ and $\varepsilon > 0$ we have $P_\delta$ such that

$$s_P(\chi_S) - \varepsilon \leq s_{P_\delta}(\chi_{S^{int}})$$

Since $s_{P_\delta}(\chi_{S^{int}}) \leq \underline{I}_R(\chi_{S^{int}})$ it follows that

$$s_P(\chi_S) - \varepsilon \leq \underline{I}_R(\chi_{S^{int}})$$

Now taking supremum over all partitions $P$ yields

$$\underline{I}_R(\chi_S) - \varepsilon \leq \underline{I}_R(\chi_{S^{int}})$$

Since $\varepsilon > 0$ was arbitrary, then $\underline{I}_R(\chi_S) \leq \underline{I}(\chi_{S^{int}})$. On the other hand $\chi_{S^{int}} \leq \chi_S$ immediately implies $\underline{I}_R(\chi_{S^{int}}) \leq \underline{I}_R(\chi_S)$.

Therefore equality follows and $\underline{I}_R(\chi_S) = \underline{I}_R(\chi_{S^{int}})$. In other words we have shown $S$ and $S^{int}$ have the same inner area.

**4.** Let $S$ be a bounded set in $\mathbb{R}^2$. Show that $S$ and $\overline{S}$ have the same outer area. (*Hint:* For any rectangle that does not intersect $S$, there are slightly smaller rectangles that do not intersect $\overline{S}$.)

Let $\overline{S} \subseteq [a,b] \times [c,d]$. By definition the outer area is

$$\overline{A} = \overline{I}_R(\chi_S) = \inf\{ S_P(f) : P \text{ is a partition of } [a,b]\times[c,d]\}$$

Here $P = \{x_0, \dots, x_J; y_0, \dots, y_K\}$ where $x_0 = a < x_1 < \cdots < x_J = b$

and $y_0 = c < y_1 < \cdots < y_K = d$.

Also $R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$ and $|R_{jk}| = (x_j - x_{j-1})(y_k - y_{k-1})$ so that

$$S_P(f) = \sum_{j,k} M_{jk} |R_{jk}| \quad \text{where} \quad M_{jk} = \sup\{ f(x,y) : (x,y) \in R_{jk}\}.$$

Setting $f(x,y) = \chi_S(x,y)$ we obtain

$$M_{jk} = \begin{cases} 1 & \text{if } R_{jk} \cap S \neq \emptyset \\ 0 & \text{if } R_{jk} \cap S = \emptyset \end{cases}$$

Given $\varepsilon > 0$ choose $\delta > 0$ so $2\delta < \min\{\Delta x_j : j = 1, \dots, J\} \cup \{\Delta y_k : k = 1, \dots, K\}$

and further so $\delta < \varepsilon / 2[K(b-a) + J(d-c)]$.

refine the partition $P$ to obtain $P_\delta$ as

$$x_0 < x_0 + \delta < x_1 - \delta < x_1 < x_1 + \delta < \cdots < x_J - \delta < x_J$$
$$y_0 < y_0 + \delta < y_1 - \delta < y_1 < y_1 + \delta < \cdots < y_K - \delta < y_K$$

Note that $2J$ points have been added to the $x$'s and $2K$ to the $y$'s

Note that

$$[x_{j-1}+\delta,\ x_j-\delta]\times[y_{k-1}+\delta,\ y_k-\delta] \subseteq (x_{j-1}, x_j)\times(y_{k-1}, y_k)$$

Therefore if $R_{jk}\cap S = \emptyset$

$$\text{then } [x_{j-1}+\delta,\ x_j-\delta]\times[y_{k-1}+\delta,\ y_k-\delta] \cap \bar{S} = \emptyset.$$

Let $R_{jk}^{\delta} = [x_{j-1}+\delta,\ x_j-\delta]\times[y_{k-1}+\delta,\ y_k-\delta]$. Then

$$|R_{jk}| - |R_{jk}^{\delta}| = 2\delta\,\Delta x_j + 2\delta\,\Delta y_j - 4\delta^2 < 2\delta(\Delta x_j + \Delta y_j).$$

Let $\bar{M}_{jk} = \sup\{\chi_{\bar{S}} : (x,y)\in R_{jk}^{\delta}\}$ then $\bar{M}_{jk} \le M_{jk}$.

Since $0 \le \chi_{\bar{S}} \le 1$ then $\bar{M}_{jk} \le 1$ and

$$S_{P_\delta}(\chi_{\bar{S}}) \le \sum_{jk} \bar{M}_{jk}\,|R_{jk}^{\delta}| + \sum_{j,k} 2\delta(\Delta x_j + \Delta y_j)$$

$$\le \sum_{j,k} M_{jk}\,|R_{jk}| + \sum_{j,k} 2\delta(\Delta x_j + \Delta y_j)$$

$$= S_P(\chi_S) + 2\delta\big(J(b-a) + K(d-c)\big) = S_P(\chi_S) + \varepsilon.$$

Since $\bar{I}_R(\chi_{\bar{S}}) \le S_{P_\delta}(\chi_{\bar{S}})$ it follows that

$$\bar{I}_R(\chi_{\bar{S}}) \le S_P(\chi_S) + \varepsilon.$$

Taking infimum on the right then yields

$$\bar{I}_R(\chi_{\bar{S}}) \le \bar{I}_R(\chi_S) + \varepsilon$$

and since $\varepsilon$ is arbitrary then

$$\overline{I}_R(\chi_{\overline{S}}) \leq \overline{I}_R(\chi_S).$$

Now $\chi_S \leq \chi_{\overline{S}}$ immediately implies $\overline{I}_R(\chi_S) \leq \overline{I}_R(\chi_{\overline{S}})$.
Therefore equality follows and $\overline{I}_R(\chi_S) = \overline{I}_R(\chi_{\overline{S}})$. In other words we have shown $S$ and $\overline{S}$ have the same outer area.

**5.** Let $S$ be a bounded set in $\mathbb{R}^2$. Show that the inner area of $S$ plus the outer area of $\partial S$ equals the outer area of $S$. (Use Exercises 3 and 4.)

Let $S \subseteq [a,b] \times [c,d]$. By definition the outer area is

$$\overline{A} = \overline{I}_R(\chi_S) = \inf\{ \overline{S}_P(f) : P \text{ is a partition of } [a,b]\times[c,d]\}$$

Here $P = \{x_0, \dots, x_J; y_0, \dots, y_K\}$ where $x_0 = a < x_1 < \dots < x_J = b$
and $y_0 = c < y_1 < \dots < y_K = d$.

Also $R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$ and $|R_{jk}| = (x_j - x_{j-1})(y_k - y_{k-1})$ so that

Define

$$\mathcal{J}_1 = \{(j,k) : R_{jk} \subseteq S^{\text{int}}\}$$

$$\mathcal{J}_2 = \{(j,k) : R_{jk} \cap \overline{S} \neq \emptyset\} \setminus \mathcal{J}_1$$

Claim that $\partial S \subseteq \bigcup_{(j,k)\in \mathcal{J}_2} R_{jk}$. Let $x \in \partial S$ since $x \in \bar{S} \subseteq [a,b]\times[c,d]$ then there is some $R_{jk}$ for which $x \in R_{jk}$. Clearly $(j,k) \notin \mathcal{J}_1$ because that would imply $x \in S^{int}$ contradicting $x \in \partial S$.

Similarly, $R_{jk} \cap \bar{S} = \emptyset$ implies $x \notin R_{jk}$.

Thus $(j,k) \in \mathcal{J}_2$ and so $x \in \bigcup_{(j,k)\in\mathcal{J}_2} R_{jk}$ proving the claim

Define

$$\bar{M}_{jk} = \sup\{\chi_{\bar{S}}(x,y) : (x,y) \in R_{jk}\}$$

$$m^{int}_{jk} = \inf\{\chi_{S^{int}}(x,y) : (x,y) \in R_{jk}\}.$$

$$M^{\partial}_{jk} = \sup\{\chi_{\partial S}(x,y) : (x,y) \in R_{jk}\}.$$

Note that $(j,k) \in \mathcal{J}_1$ implies $\bar{M}_{jk} = 1 = m^{int}_{jk}$ and $M^{\partial}_{jk} = 0$

Also $(j,k) \in \mathcal{J}_2$ implies $M^{\partial}_{jk} = 1 = \bar{M}_{jk}$ and $m^{int}_{jk} = 0$

Finally $(j,k) \notin \mathcal{J}_1 \cup \mathcal{J}_2$ implies $\bar{M}_{jk} = M^{\partial}_{jk} = 0 = m^{int}_{jk}$. Thus,

$$S_P(\chi_{\bar{S}}) = \sum_{j,k} \bar{M}_{jk}|R_{jk}| = \sum_{(j,k)\in\mathcal{J}_1} \bar{M}_{jk}|R_{jk}| + \sum_{(j,k)\in\mathcal{J}_2} \bar{M}_{jk}|R_{jk}|$$

$$= \sum_{(j,k)\in\mathcal{J}_1} m^{int}_{jk}|R_{jk}| + \sum_{(j,k)\in\mathcal{J}_2} M^{\partial}_{jk}|R^{\partial}_{jk}|$$

$$= \sum_{j,k} m^{int}_{jk}|R_{jk}| + \sum_{j,k} M^{\partial}_{jk}|R^{\partial}_{jk}| = s_P(\chi_{S^{int}}) + S_P(\chi_{\partial S})$$

Taking limits yields that

$$\overline{I}_R(\chi_{\bar{S}}) = \underline{I}_R(\chi_{S^{int}}) + \overline{I}_R(\chi_{\partial S})$$

Finally, since the previous 2 questions imply

$$\overline{I}_R(\chi_{\bar{S}}) = \overline{I}_R(\chi_S) \quad \text{and} \quad \underline{I}_R(\chi_{S^{int}}) = \underline{I}_R(\chi_S)$$

we obtain

$$\overline{I}_R(\chi_S) = \underline{I}_R(\chi_S) + \overline{I}_R(\chi_{\partial S}).$$

In otherwords

$$\overline{A}(S) = \underline{A}(S) + \overline{A}(\partial S).$$

**7.** (*The Second Mean Value Theorem for Integrals*) Suppose $f$ is continuous on $[a,b]$ and $\varphi$ is of class $C^1$ and increasing on $[a,b]$. Show that there is a point $c \in [a,b]$ such that

$$\int_a^b f(x)\varphi(x)\,dx = \varphi(a)\int_a^c f(x)\,dx + \varphi(b)\int_c^b f(x)\,dx.$$

(*Hint:* First suppose $\varphi(b) = 0$. Set $F(x) = \int_a^x f(t)\,dt$, integrate by parts to show that $\int_a^b f(x)\varphi(x)\,dx = -\int_a^b F(x)\varphi'(x)\,dx$, and apply Theorem 4.24 to the latter integral. To remove the condition $\varphi(b) = 0$, show that if the conclusion is true for $f$ and $\varphi$, it is true for $f$ and $\varphi + C$ for any constant $C$.)

Recall

**4.24 Theorem** (The Mean Value Theorem for Integrals). *Let $S$ be a compact, connected, measurable susbset of $\mathbb{R}^n$, and let $f$ and $g$ be continuous functions on $S$ with $g \geq 0$. Then there is a point $\mathbf{a} \in S$ such that*

$$\int \cdots \int_S f(\mathbf{x})g(\mathbf{x})\, d^n\mathbf{x} = f(\mathbf{a}) \int \cdots \int_S g(\mathbf{x})\, d^n\mathbf{x}.$$

Define $F(x) = \int_a^x f(t)\,dt$. Then integrating by parts as

$$\int_a^b f(x)\varphi(x)\,dx = \varphi(x)F(x)\Big|_a^b - \int_a^b F(x)\varphi'(x)\,dx$$

$$u = \varphi(x) \qquad du = \varphi'(x)\,dx$$

$$dv = f(x)\,dx \qquad v = F(x)$$

so

$$\int_a^b f(x)\varphi(x)\,dx = \varphi(b)F(b) - \int_a^b F(x)\varphi'(x)\,dx$$

Since $\varphi$ is increasing then $\varphi'(x) \geq 0$. Note also that $F$ is continuous.

Therefore, by Theorem 4.24 there is a point $c$ such that

$$\int_a^b F(x)\varphi'(x)\,dx = F(c)\int_a^b \varphi'(x)\,dx = F(c)\big(\varphi(b) - \varphi(a)\big)$$

It follows that

$$\int_a^b f(x)\varphi(x)\,dx = \varphi(b)\big(F(b) - F(c)\big) + \varphi(a)F(c)$$

$$= \varphi(b)\left(\int_a^b f(t)\,dt - \int_a^c f(t)\,dt\right) + \varphi(a)\int_a^c f(t)\,dt$$

$$= \varphi(a)\int_a^c f(t)\,dt + \varphi(b)\int_c^b f(t)\,dt.$$

**5.** Evaluate the following iterated integrals. (You may need to reverse the order of integration.)

a. $\int_1^3 \int_1^y y e^{2x}\, dx\, dy$.

b. $\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3+1)\, dy\, dx$.

c. $\int_1^2 \int_{1/x}^1 y e^{xy}\, dy\, dx$.

(a) $\int_1^3 \int_1^y y e^{2x}\, dx\, dy = \int_1^3 \left( \frac{y}{2} e^{2x} \Big|_{x=1}^{y} \right) dy = \int_1^3 \frac{y}{2}\left(e^{2y} - e^2\right) dy$

$\int_1^3 \frac{y}{2} e^{2y}\, dy = \int_1^3 \frac{y}{4}\, de^{2y} = \frac{y}{4} e^{2y}\Big|_1^3 - \int_1^3 \frac{e^{2y}}{4}\, dy$

$= \frac{3}{4}e^6 - \frac{1}{4}e^2 - \frac{e^{2y}}{8}\Big|_1^3 = \frac{3}{4}e^6 - \frac{1}{4}e^2 - \frac{1}{8}e^6 + \frac{1}{8}e^2$

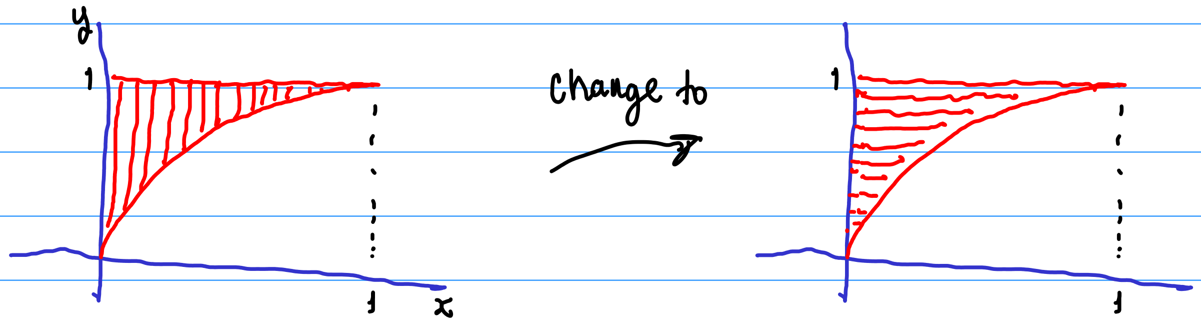
$= \frac{5}{8}e^6 - \frac{1}{8}e^2$

$-\int_1^3 \frac{y}{2} e^2\, dy = -\frac{y^2}{4} e^2 \Big|_1^3 = -\frac{9}{4}e^2 + \frac{1}{4}e^2 = -2e^2$

Therefore

$\int_1^3 \int_1^y y e^{2x}\, dx\, dy = \frac{5}{8}e^6 - \frac{17}{8}e^2$

(b) $\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3+1)\, dy\, dx = \int_0^1 \int_0^{y^2} \cos(y^3+1)\, dx\, dy$



$$\int_0^1\int_{\sqrt{x}}^1 \cos(y^3+1)\,dy\,dx = \int_0^1 x\cos(y^3+1)\Big|_{x=0}^{y^2} dy$$

$$= \int_0^1 y^2\cos(y^3+1)\,dy = \frac{1}{3}\sin(y^3+1)\Big|_0^1 = \frac{1}{3}(\sin 2 - \sin 1)$$

(c) $\int_1^2\int_{1/x}^1 ye^{xy}\,dy\,dx$

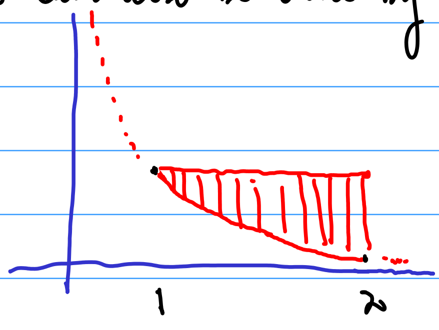
$$\int_{1/x}^1 ye^{xy}\,dy = \int_{1/x}^1 \frac{y}{x}\,de^{xy} = \frac{y}{x}e^{xy}\Big|_{y=1/x}^1 - \int_{1/x}^1 \frac{e^{xy}}{x}\,dy$$

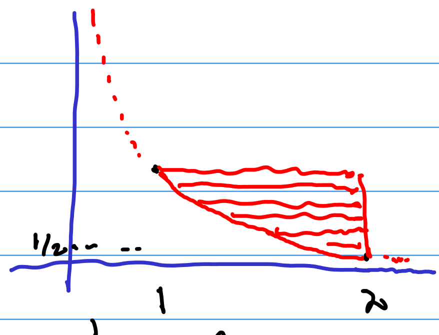
$$= \frac{1}{x}e^{x} - \frac{1}{x^2}e - \frac{1}{x^2}e^{xy}\Big|_{y=1/x}^1 = \frac{1}{x}e^{x} - \frac{1}{x^2}e - \frac{1}{x^2}e^{x} + \frac{1}{x^2}e = \left(\frac{1}{x}-\frac{1}{x^2}\right)e^{x}$$

$$\int_1^2\int_{1/x}^1 ye^{xy}\,dy\,dx = \int_1^2\left(\frac{1}{x}-\frac{1}{x^2}\right)e^{x}\,dx = \frac{1}{x}e^{x}\Big|_1^2 = \frac{1}{2}e^2 - e$$

since $\frac{d}{dx}\left(\frac{1}{x}e^{x}\right) = -\frac{1}{x^2}e^{x} + \frac{1}{x}e^{x}$

This can also be done by reversing the order as




$$\int_1^2\int_{1/x}^1 ye^{xy}\,dy\,dx = \int_{1/2}^1\int_{1/y}^2 ye^{xy}\,dx\,dy = \int_{1/2}^1 e^{xy}\Big|_{x=1/y}^2 dy$$

$$= \int_{1/2}^1 (e^{2y}-e)\,dy = \left(\frac{1}{2}e^{2y} - ye\right)\Big|_{1/2}^1 = \frac{1}{2}e^2 - e - \frac{1}{2}e + \frac{1}{2}e = \frac{1}{2}e^2 - e.$$