powers

December 2, 2020

Matrix Powers using Eigenvectors

The matrix S is has the eigenvectors of A as columns while D is a diagonal matrix with the corresponding eigenvalues of A on the diagonal.

[23]:
```
S=[1 0 1; 0 1 1; 1 1 0]
```

[23]:
```
3×3 Array{Int64,2}:
 1  0  1
 0  1  1
 1  1  0
```

[24]:
```
using LinearAlgebra
```

[25]:
```
D=diagm([2,4,6])
```

[25]:
```
3×3 Array{Int64,2}:
 2  0  0
 0  4  0
 0  0  6
```

[26]:
```
S*D*inv(S)
```

[26]:
```
3×3 Array{Float64,2}:
  4.0  2.0  -2.0
  1.0  5.0  -1.0
 -1.0  1.0   3.0
```

[27]:
```
A=[4 2 -2; 1 5 -1; -1 1 3]
```

[27]:
```
3×3 Array{Int64,2}:
  4  2  -2
  1  5  -1
 -1  1   3
```

[28]:
```
A^2
```

[28]:
```
3×3 Array{Int64,2}:
 20  16  -16
 10  26  -10
 -6   6   10
```

[29]:
```
S*D^2*inv(S)
```

[29]:
```
3×3 Array{Float64,2}:
 20.0  16.0  -16.0
 10.0  26.0  -10.0
 -6.0   6.0   10.0
```

Rescale the matrix A by 6 and D by 6 to obtain new matrices. These matrices are designed so that the leading term growth factor of 6^n which appeared in D is divided out and one can take the limit

$$\lim_{n\to\infty}\left(\frac{A}{6}\right)^n = S\lim_{n\to\infty}\left(\frac{D}{6}\right)^n S^{-1} = S\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} S^{-1}$$

since

$$\left(\frac{D}{6}\right)^n = \begin{bmatrix}(1/3)^n & 0 & 0\\ 0 & (2/3)^n & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Let's check how this works.

[30]:
```
newA=A/6
```

[30]:
```
3×3 Array{Float64,2}:
  0.666667  0.333333  -0.333333
  0.166667  0.833333  -0.166667
 -0.166667  0.166667   0.5
```

[31]:
```
newD=D/6
```

[31]:
```
3×3 Array{Float64,2}:
 0.333333  0.0       0.0
 0.0       0.666667  0.0
 0.0       0.0       1.0
```

For convenience define matrices `newA` and `newD` to be the matrices that were rescaled by the factor 6.

[32]:
```
S*newD*inv(S)
```

[32]:
```
3×3 Array{Float64,2}:
  0.666667  0.333333  -0.333333
  0.166667  0.833333  -0.166667
 -0.166667  0.166667   0.5
```

[33]:
```
newD^100
```

[33]:
```
3×3 Array{Float64,2}:
 1.94033e-48  0.0          0.0
 0.0          2.45965e-18  0.0
 0.0          0.0          1.0
```

See how powers of $(D/6)^n$ remain diagonal and the first two terms on the diagonal tend to zero. It's clear what the limit matrix is, because we divided out by the largest eigenvalue.

For convenience let `limitD` be the limit such that

$(D/6)^n \to$ `limitD` as $n \to \infty$.

[34]:
```
limitD=[0 0 0; 0 0 0; 0 0 1]
```

[34]:
```
3×3 Array{Int64,2}:
 0  0  0
 0  0  0
 0  0  1
```

[35]:
```
S*limitD*inv(S)
```

[35]:
```
3×3 Array{Float64,2}:
 0.5  0.5  -0.5
 0.5  0.5  -0.5
 0.0  0.0   0.0
```

[36]:
```
newA^10
```

[36]:
```
3×3 Array{Float64,2}:
  0.500008   0.499992   -0.499992
  0.491329   0.508671   -0.491329
 -0.0086623  0.0086623   0.00867923
```

Note that computing $(A/6)^{10}$ is close to the limiting value, but involved much more work than computing S `limitD` S^{-1} and only approximates the limiting value of $(A/6)^n$ as $n \to \infty$.

This is also a way to take limits!

[ ]:
```

```