Math 330 Linear Algebra Homework 10

9.3 *Equilibrium point for linear dynamical system.* Consider a time-invariant linear dynamical system with offset, $x_{t+1} = Ax_t + c$, where x_t is the state n-vector. We say that a vector z is an *equilibrium point* of the linear dynamical system if $x_1 = z$ implies $x_2 = z$, $x_3 = z$, ... (In words: If the system starts in state z, it stays in state z.)

Find a matrix F and vector g for which the set of linear equations $Fz = g$ characterizes equilibrium points. (This means: If z is an equilibrium point, then $Fz = g$; conversely if $Fz = g$, then z is an equilibrium point.) Express F and g in terms of A, c, any standard matrices or vectors (*e.g.*, I, $\mathbf{1}$, or 0), and matrix and vector operations.

Remark. Equilibrium points often have interesting interpretations. For example, if the linear dynamical system describes the population dynamics of a country, with the vector c denoting immigration (emigration when entries of c are negative), an equilibrium point is a population distribution that does not change, year to year. In other words, immigration exactly cancels the changes in population distribution caused by aging, births, and deaths.

If z is an equilibrium point, then $x_i = z$ for all values of i. In particular

$$x_{t+1} = Ax_t + c$$

implies

$$z = Az + c$$

Thus

$$z - Az = c \quad \text{or} \quad (I - A)z = c.$$

Taking $F = I - A$ and $g = c$ makes this condition equivalent to $Fz = g$. Thus $Fz = g$ means that z is an equilibrium point and if z is an equilibrium point then $Fz = g$.

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9.2 *Dynamics of an economy.* An economy (of a country or region) is described by an n-vector a_t, where $(a_t)_i$ is the economic output in sector i in year t (measured in billions of dollars, say). The total output of the economy in year t is $\mathbf{1}^T a_t$. A very simple model of how the economic output changes over time is $a_{t+1} = Ba_t$, where B is an $n \times n$ matrix. (This is closely related to the Leontief input-output model described on page 157 of the book. But the Leontief model is static, *i.e.*, doesn't consider how an economy changes over time.) The entries of a_t and B are positive in general.

In this problem we will consider the specific model with $n = 4$ sectors and

$$B = \begin{bmatrix} 0.10 & 0.06 & 0.05 & 0.70 \\ 0.48 & 0.44 & 0.10 & 0.04 \\ 0.00 & 0.55 & 0.52 & 0.04 \\ 0.04 & 0.01 & 0.42 & 0.51 \end{bmatrix}.$$

(a) Briefly interpret B_{23}, in English.

(b) *Simulation.* Suppose $a_1 = (0.6, 0.9, 1.3, 0.5)$. Plot the four sector outputs (*i.e.*, $(a_t)_i$ for $i = 1, \ldots, 4$) and the total economic output (*i.e.*, $\mathbf{1}^T a_t$) versus t, for $t = 1, \ldots, 20$.

(a) B_{23} is the amount that sector 3 contributes to the growth of sector 2 per year.

(b) In julia on the next page

hw10p92b

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Homework 10 Question 9.2 Part (b)

[1]:
```
B=[0.10 0.06 0.05 0.70;
   0.48 0.44 0.10 0.04;
   0.00 0.55 0.52 0.04;
   0.04 0.01 0.42 0.51]
```

[1]:
```
4×4 Array{Float64,2}:
 0.1   0.06  0.05  0.7
 0.48  0.44  0.1   0.04
 0.0   0.55  0.52  0.04
 0.04  0.01  0.42  0.51
```

[2]:
```
A=zeros(4,20);
```

[3]:
```
A[:,1]=[0.6,0.9,1.3,0.5]
```

[3]:
```
4-element Array{Float64,1}:
 0.6
 0.9
 1.3
 0.5
```

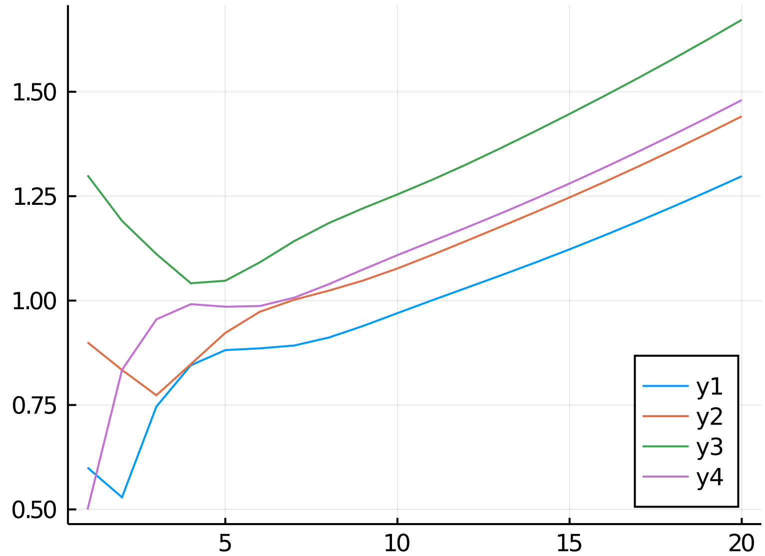
[4]:
```
for t=1:19
    A[:,t+1]=B*A[:,t]
end
```

[5]:
```
using Plots
```

The plot of the four sector outputs

[6]:
```
plot(A',size=[400,300],legend=:bottomright)
```

[6]:

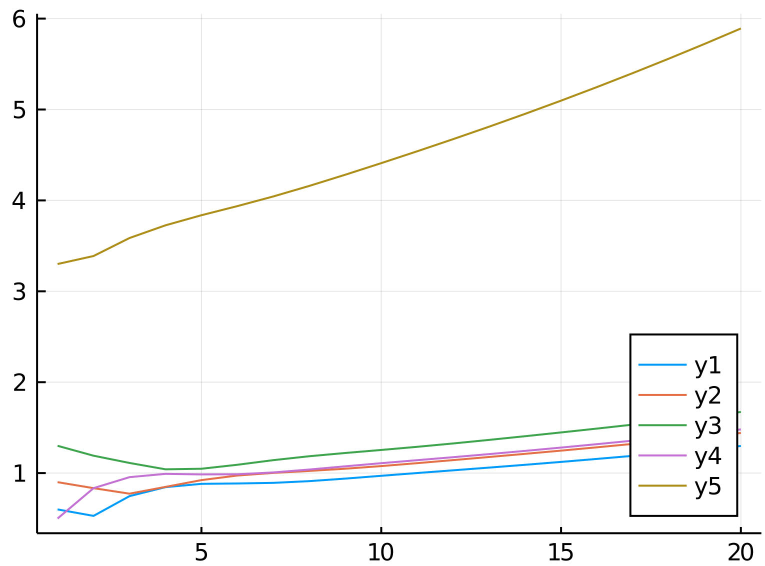


[7]:
```
totals=A'*[1,1,1,1];
```

Overlay the total economic output on the previous graph

[8]:
```
plot!(totals)
```

[8]:



[ ]:
```

```