Math 330: Quiz 3 Version 1

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Work each problem using pencil and paper or using a computer and Julia as appropriate. Include all work, programs and computer output used to arrive at your final answers. When you are finished upload a high-resolution version of your work as a single PDF file for grading to WebCampus.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Consider the block matrix

$$A = \begin{bmatrix} I & B & 0 \\ B^T & 0 & 0 \\ 0 & 0 & BB^T \end{bmatrix},$$

where B is 5×4.

(i) What are the dimensions of the four zero matrices and the identity matrix in the definition of A?

(ii) What are the dimensions of A?

3. Let $a_1, \ldots, a_n$ be the columns of the $m \times n$ matrix A. Suppose that the columns all have norm one, and for $i \neq j$ that $\angle(a_i, a_j) = 52°$.

(i) What can you say about the Gram matrix $G = A^T A$? Be as specific as possible.

(ii) [Extra Credit] Construct a concrete example of such an A with $m \geq 3$ and $n \geq 3$.

4. The number 1 has two square roots 1 and -1. The $n \times n$ identity matrix I_7 has many more square roots.

(i) Find all diagonal square roots of I_7. How many are there?

(ii) Find a nondiagonal 2×2 matrix A that satisfies $A^2 = I$. This means that in general there are even more square roots of I_7 that you found in part (i).

5. Consider the 6×6 matrix

$$A = \begin{bmatrix} I_5 & a \\ a^T & 0 \end{bmatrix}$$

where a is an 5-vector.

(i) When is A invertible? Give your answer in terms of a. Justify your answer.

(ii) Assuming the condition you found in part (i) holds, give an expression for the inverse matrix A^{-1}.

6. Consider the matrix A and vector b given by

$$A = \begin{bmatrix} 8 & -2 & 7 & -2 \\ -6 & 8 & 1 & 4 \\ 1 & 8 & -6 & 3 \\ -1 & -5 & -9 & 2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 8 \\ 5 \\ -2 \\ -1 \end{bmatrix}.$$

(i) Find a matrix Q with orthonormal columns and an upper triagular matrix R such that $A = QR$. If you use Julia for this (recommended) then include the commands you typed as well as the output as part of your answer.

(ii) Given your answer for Q above calculate $y = Q^T b$.

(iii) Given your answer for R and y above solve $Rx = y$ for x.

(iv) Calculate $Ax - b$ and explain why this value should be close to zero.

7. Given a 8×9 matrix A with linearly independent rows define its pseudo-inverse by $A^\dagger = A^T(AA^T)^{-1}$. Show that

$$A = AA^\dagger A \qquad \text{and} \qquad A^\dagger = A^\dagger A A^\dagger.$$

8. Consider the 100×100 population dynamics matrix

$$A = \begin{bmatrix} b_1 & b_2 & \cdots & b_{99} & b_{100} \\ 1-d_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-d_{99} & 0 \end{bmatrix},$$

where $b_i \geq 0$ are the birth rates and $0 \leq d_i \leq 1$ are death rates. What are the conditions on b_i and d_i under which A is invertible? (If the matrix is never invertible or always invertible, say so.) Justify your answer.

Math 330: Quiz 3 Version 2

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1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Consider the block matrix

$$A = \begin{bmatrix} I & B & 0 \\ B^T & 0 & 0 \\ 0 & 0 & BB^T \end{bmatrix},$$

where B is 5×4.

(i) What are the dimensions of the four zero matrices and the identity matrix in the definition of A?

(ii) What are the dimensions of A?

3. Let $a_1, \ldots, a_n$ be the columns of the $m \times n$ matrix A. Suppose that the columns all have norm one, and for $i \neq j$ that $\angle(a_i, a_j) = 75°$.

(i) What can you say about the Gram matrix $G = A^T A$? Be as specific as possible.

(ii) [Extra Credit] Construct a concrete example of such an A with $m \geq 3$ and $n \geq 3$.

4. The number 1 has two square roots 1 and -1. The $n \times n$ identity matrix I_6 has many more square roots.

(i) Find all diagonal square roots of I_6. How many are there?

(ii) Find a nondiagonal 2×2 matrix A that satisfies $A^2 = I$. This means that in general there are even more square roots of I_6 that you found in part (i).

5. Consider the 10×10 matrix

$$A = \begin{bmatrix} I_9 & a \\ a^T & 0 \end{bmatrix}$$

where a is an 9-vector.

(i) When is A invertible? Give your answer in terms of a. Justify your answer.

(ii) Assuming the condition you found in part (i) holds, give an expression for the inverse matrix A^{-1}.

6. Consider the matrix A and vector b given by

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -6 & 2 & 7 & 6 \\ 7 & -5 & -7 & 7 \\ -6 & 6 & 0 & -8 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -4 \end{bmatrix}.$$

(i) Find a matrix Q with orthonormal columns and an upper triagular matrix R such that $A = QR$. If you use Julia for this (recommended) then include the commands you typed as well as the output as part of your answer.

(ii) Given your answer for Q above calculate $y = Q^T b$.

(iii) Given your answer for R and y above solve $Rx = y$ for x.

(iv) Calculate $Ax - b$ and explain why this value should be close to zero.

7. Given a 6×8 matrix A with linearly independent rows define its pseudo-inverse by $A^\dagger = A^T(AA^T)^{-1}$. Show that

$$A = AA^\dagger A \qquad \text{and} \qquad A^\dagger = A^\dagger A A^\dagger.$$

8. Consider the 100×100 population dynamics matrix

$$A = \begin{bmatrix} b_1 & b_2 & \cdots & b_{99} & b_{100} \\ 1-d_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-d_{99} & 0 \end{bmatrix},$$

where $b_i \geq 0$ are the birth rates and $0 \leq d_i \leq 1$ are death rates. What are the conditions on b_i and d_i under which A is invertible? (If the matrix is never invertible or always invertible, say so.) Justify your answer.

Math 330: Quiz 3 Version 3

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1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Consider the block matrix

$$A = \begin{bmatrix} I & B & 0 \\ B^T & 0 & 0 \\ 0 & 0 & BB^T \end{bmatrix},$$

where B is 4×7.

(i) What are the dimensions of the four zero matrices and the identity matrix in the definition of A?

(ii) What are the dimensions of A?

3. Let $a_1, \ldots, a_n$ be the columns of the $m \times n$ matrix A. Suppose that the columns all have norm one, and for $i \neq j$ that $\angle(a_i, a_j) = 33°$.

(i) What can you say about the Gram matrix $G = A^T A$? Be as specific as possible.

(ii) [Extra Credit] Construct a concrete example of such an A with $m \geq 3$ and $n \geq 3$.

4. The number 1 has two square roots 1 and -1. The $n \times n$ identity matrix I_6 has many more square roots.

(i) Find all diagonal square roots of I_6. How many are there?

(ii) Find a nondiagonal 2×2 matrix A that satisfies $A^2 = I$. This means that in general there are even more square roots of I_6 that you found in part (i).

5. Consider the 9×9 matrix

$$A = \begin{bmatrix} I_8 & a \\ a^T & 0 \end{bmatrix}$$

where a is an 8-vector.

(i) When is A invertible? Give your answer in terms of a. Justify your answer.

(ii) Assuming the condition you found in part (i) holds, give an expression for the inverse matrix A^{-1}.

6. Consider the matrix A and vector b given by

$$A = \begin{bmatrix} -7 & 1 & -2 & 8 \\ -8 & 4 & 9 & -8 \\ -6 & 5 & 5 & 4 \\ -2 & 7 & -3 & -5 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} -5 \\ -8 \\ -8 \\ -9 \end{bmatrix}.$$

(i) Find a matrix Q with orthonormal columns and an upper triagular matrix R such that $A = QR$. If you use Julia for this (recommended) then include the commands you typed as well as the output as part of your answer.

(ii) Given your answer for Q above calculate $y = Q^T b$.

(iii) Given your answer for R and y above solve $Rx = y$ for x.

(iv) Calculate $Ax - b$ and explain why this value should be close to zero.

7. Given a 8×9 matrix A with linearly independent rows define its pseudo-inverse by $A^\dagger = A^T(AA^T)^{-1}$. Show that

$$A = AA^\dagger A \qquad \text{and} \qquad A^\dagger = A^\dagger AA^\dagger.$$

8. Consider the 100×100 population dynamics matrix

$$A = \begin{bmatrix} b_1 & b_2 & \cdots & b_{99} & b_{100} \\ 1-d_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-d_{99} & 0 \end{bmatrix},$$

where $b_i \geq 0$ are the birth rates and $0 \leq d_i \leq 1$ are death rates. What are the conditions on b_i and d_i under which A is invertible? (If the matrix is never invertible or always invertible, say so.) Justify your answer.

Math 330: Quiz 3 Version 4

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1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Consider the block matrix

$$A = \begin{bmatrix} I & B & 0 \\ B^T & 0 & 0 \\ 0 & 0 & BB^T \end{bmatrix},$$

where B is 5×7.

(i) What are the dimensions of the four zero matrices and the identity matrix in the definition of A?

(ii) What are the dimensions of A?

3. Let $a_1, \ldots, a_n$ be the columns of the $m \times n$ matrix A. Suppose that the columns all have norm one, and for $i \neq j$ that $\angle(a_i, a_j) = 33°$.

(i) What can you say about the Gram matrix $G = A^T A$? Be as specific as possible.

(ii) [Extra Credit] Construct a concrete example of such an A with $m \geq 3$ and $n \geq 3$.

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5. Consider the 6×6 matrix

$$A = \begin{bmatrix} I_5 & a \\ a^T & 0 \end{bmatrix}$$

where a is an 5-vector.

(i) When is A invertible? Give your answer in terms of a. Justify your answer.

(ii) Assuming the condition you found in part (i) holds, give an expression for the inverse matrix A^{-1}.

6. Consider the matrix A and vector b given by

$$A = \begin{bmatrix} 5 & -2 & -5 & -3 \\ 9 & -2 & 9 & 6 \\ 5 & -8 & -2 & -6 \\ 9 & 9 & 0 & -2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 0 \\ 2 \\ -7 \\ -4 \end{bmatrix}.$$

(i) Find a matrix Q with orthonormal columns and an upper triagular matrix R such that $A = QR$. If you use Julia for this (recommended) then include the commands you typed as well as the output as part of your answer.

(ii) Given your answer for Q above calculate $y = Q^T b$.

(iii) Given your answer for R and y above solve $Rx = y$ for x.

(iv) Calculate $Ax - b$ and explain why this value should be close to zero.

7. Given a 9×7 matrix A with linearly independent columns define its pseudo-inverse by $A^\dagger = (A^T A)^{-1} A^T$. Show that

$$A = AA^\dagger A \qquad \text{and} \qquad A^\dagger = A^\dagger A A^\dagger.$$

8. Consider the 100×100 population dynamics matrix

$$A = \begin{bmatrix} b_1 & b_2 & \cdots & b_{99} & b_{100} \\ 1-d_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-d_{99} & 0 \end{bmatrix},$$

where $b_i \geq 0$ are the birth rates and $0 \leq d_i \leq 1$ are death rates. What are the conditions on b_i and d_i under which A is invertible? (If the matrix is never invertible or always invertible, say so.) Justify your answer.

1. I have worked independently on this quiz.

— Test Student

2. <u>Version 1&2:</u> Let $B \in \mathbb{R}^{5\times 4}$

Then

$$A = \begin{array}{c|c} & \begin{array}{ccc} \quad 5 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\ \hline \begin{array}{c} 5 \\[2ex] 4 \\[2ex] 5 \end{array} & \begin{bmatrix} \underset{5\times 5}{I} & \underset{5\times 4}{B} & \underset{5\times 5}{0} \\[2ex] \underset{4\times 5}{B^T} & \underset{4\times 4}{0} & \underset{4\times 5}{0} \\[2ex] \underset{5\times 5}{0} & \underset{5\times 4}{0} & \underset{5\times\cancel{4}\ \cancel{4}\times 5}{BB^T} \end{bmatrix} \end{array} \in \mathbb{R}^{14\times 14}$$

(i) (ii)

<u>Version 3:</u> Let $B \in \mathbb{R}^{4\times 7}$

Then

$$A = \begin{array}{c|c} & \begin{array}{ccc} \quad 4 \quad & \quad 7 \quad & \quad 4 \quad \end{array} \\ \hline \begin{array}{c} 4 \\[2ex] 7 \\[2ex] 4 \end{array} & \begin{bmatrix} \underset{4\times 4}{I} & \underset{4\times 7}{B} & \underset{4\times 4}{0} \\[2ex] \underset{7\times 4}{B^T} & \underset{7\times 7}{0} & \underset{7\times 4}{0} \\[2ex] \underset{4\times 4}{0} & \underset{4\times 7}{0} & \underset{4\times\cancel{7}\ \cancel{7}\times 4}{BB^T} \end{bmatrix} \end{array} \in \mathbb{R}^{15\times 15}$$

(i) (ii)

2. Version 4: Let $B \in \mathbb{R}^{5\times 7}$

Then

$$A = \begin{array}{cc} & \begin{matrix} 5 & \quad 7 & \quad 5 \end{matrix} \\ \begin{matrix} 5 \\ 7 \\ 5 \end{matrix} & \begin{bmatrix} \underset{5\times5}{I} & \underset{5\times7}{B} & \underset{5\times5}{0} \\ \underset{7\times5}{B^T} & \underset{7\times7}{0} & \underset{7\times5}{0} \\ \underset{5\times5}{0} & \underset{5\times7}{0} & \underset{5\times\cancel{7}\,\cancel{7}\times5}{BB^T} \end{bmatrix} \end{array} \in \mathbb{R}^{17\times17}$$

(i) (ii)

3. Version 1: $\angle(a_i, a_j) = 52^\circ$ for $i \neq j$.

$$a_i \cdot a_j = \begin{cases} 1 & \text{for } i = j \\ \cos 52^\circ & \text{for } i \neq j \end{cases}$$

(i)

$$A^TA = \begin{bmatrix} 1 & \cos52^\circ & \cdots & \cos52^\circ \\ \cos52^\circ & 1 & \cos52^\circ & \vdots \\ \vdots & \cos52^\circ & 1 & \ddots \\ & \ddots & 1 & \cos52^\circ \\ \cos52^\circ & \cdots & \cos52^\circ & 1 \end{bmatrix}$$

The Gram matrix has 1 on the diagonal and $\cos 52^\circ \approx 0.6156614753256583\ldots$ for all off-diagonal terms.

3. Version 2: $\angle(a_i, a_j) = 75°$ for $i \neq j$.

The Gram matrix has 1 on the diagonal

(i) and $\cos 75° \approx 0.25881904102520 74\ldots$ for all off-diagonal entries.

Version 3&4 $\angle(a_i, a_j) = 33°$ for $i \neq j$

(i) ~~----------~~

~~#~~ $\cos 33° \approx 0.838670567945424\ldots$ for

~~----------~~ .

3. (ii) Extra Credit: $\angle(a_i, a_j) = \theta$ for $i \neq j$

Let $a_1 = (1, 0, 0, 0)$

Now take $a_2 = (\alpha, \beta, 0, 0)$ and solve so that $a_1^T a_2 = \alpha = \cos\theta$

and $a_2^T a_2^T = \alpha^2 + \beta^2 = 1$.

It follows that $\beta = \sqrt{1 - \cos^2\theta} = \sin\theta$ and

$$a_2 = (\cos\theta, \sin\theta, 0, 0)$$

Now take $a_3 = (\cos\theta, \gamma, \delta, 0)$ and solve so $a_2^T a_3 = \cos^2\theta + \sin\theta\,\gamma = \cos\theta$

and $a_3^T a_3 = \cos^2\theta + \gamma^2 + \delta^2 = 1$.

3. continuess...

It follows that

$$\gamma = \frac{\cos\vartheta - \cos^2\vartheta}{\sin\vartheta}$$

and

$$\delta = \sqrt{1-\cos^2\vartheta - \left(\frac{\cos\vartheta - \cos^2\vartheta}{\sin\vartheta}\right)^2}$$

$$= \sqrt{(1-\cos\vartheta)(1+\cos\vartheta) - \frac{(1-\cos\vartheta)^2\cos^2\vartheta}{\sin^2\vartheta}}$$

$$= \sqrt{\frac{(1-\cos\vartheta)^{\cancel{2}}(1+\cos\vartheta)^2 - (1-\cos\vartheta)^{\cancel{2}}\cos^2\vartheta}{\cancel{(1-\cos\vartheta)}(1+\cos\vartheta)}}$$

$$= \sqrt{\frac{(1-\cos\vartheta)(1+\cos\vartheta)^2 - (1-\cos\vartheta)\cos^2\vartheta}{1+\cos\vartheta}}$$

$$= \sqrt{\frac{(1-\cos\vartheta)(1+2\cos\vartheta + \cancel{\cos^2\vartheta} - \cancel{\cos^2\vartheta})}{1+\cos\vartheta}}$$

$$= \sqrt{\frac{(1-\cos\vartheta)(1+2\cos\vartheta)}{1+\cos\vartheta}}$$

Since $0 < \cos\theta < 1$ then the fraction

$$\frac{(1-\cos\theta)(1+2\cos\theta)}{1+\cos\theta} > 0$$

and so the square root in the definition of δ makes sense. In particular δ really exists and is a real number.

We, therefore have the matrix

$$A = \begin{bmatrix} 1 & \cos\theta & \cos\theta \\ 0 & \sin\theta & \frac{\cos\theta - \cos^2\theta}{\sin\theta} \\ 0 & 0 & \sqrt{\frac{(1-\cos\theta)(1+2\cos\theta)}{1+\cos\theta}} \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 3}$$

Examples in Julian when

$\theta = 52^\circ$

$\theta = 75^\circ$

and $\theta = 33^\circ$ are ...

q3iiv1

November 29, 2020

Question 3(ii) Extra Credit Version 1

[1]:
```
theta=52/360*2*pi
```

[1]:
```
0.9075712110370513
```

[2]:
```
a1=[1,0,0,0]
```

[2]:
```
4-element Array{Int64,1}:
 1
 0
 0
 0
```

[3]:
```
a2=[cos(theta),sin(theta),0,0]
```

[3]:
```
4-element Array{Float64,1}:
 0.6156614753256583
 0.7880107536067219
 0.0
 0.0
```

[5]:
```
a3=[cos(theta),cos(theta)*(1-cos(theta))/sin(theta),
    sqrt((1-cos(theta))*(1+2*cos(theta))/(1+cos(theta))),0]
```

[5]:
```
4-element Array{Float64,1}:
 0.6156614753256583
 0.3002781650408606
 0.7285560866532703
 0.0
```

[6]:
```
A=[a1 a2 a3]
```

[6]:
```
4×3 Array{Float64,2}:
 1.0  0.615661  0.615661
 0.0  0.788011  0.300278
 0.0  0.0       0.728556
 0.0  0.0       0.0
```

[7]:

```
A'*A
```

[7]:

```
3×3 Array{Float64,2}:
 1.0       0.615661  0.615661
 0.615661  1.0       0.615661
 0.615661  0.615661  1.0
```

[ ]:

```

```

q3iiv2

November 29, 2020

Question 3(ii) Extra Credit Version 2

[8]:
```
theta=75/360*2*pi
```

[8]:
```
1.3089969389957472
```

[9]:
```
a1=[1,0,0,0]
```

[9]:
```
4-element Array{Int64,1}:
 1
 0
 0
 0
```

[10]:
```
a2=[cos(theta),sin(theta),0,0]
```

[10]:
```
4-element Array{Float64,1}:
 0.25881904510252074
 0.9659258262890683
 0.0
 0.0
```

[11]:
```
a3=[cos(theta),cos(theta)*(1-cos(theta))/sin(theta),
    sqrt((1-cos(theta))*(1+2*cos(theta))/(1+cos(theta))),0]
```

[11]:
```
4-element Array{Float64,1}:
 0.25881904510252074
 0.1985988383101079
 0.9452889522860695
 0.0
```

[12]:
```
A=[a1 a2 a3]
```

[12]:
```
4×3 Array{Float64,2}:
 1.0  0.258819  0.258819
 0.0  0.965926  0.198599
 0.0  0.0       0.945289
 0.0  0.0       0.0
```

[13]:
```
A'*A
```

[13]:
```
3×3 Array{Float64,2}:
 1.0       0.258819  0.258819
 0.258819  1.0       0.258819
 0.258819  0.258819  1.0
```

[ ]:
```
```

q3iiv34

November 29, 2020

Question 3(ii) Extra Credit Versions 3 and 4

[14]:
```
theta=33/360*2*pi
```

[14]:
```
0.5759586531581287
```

[20]:
```
a1=[1,0,0,0];
```

[21]:
```
a2=[cos(theta),sin(theta),0,0];
```

[22]:
```
a3=[cos(theta),cos(theta)*(1-cos(theta))/sin(theta),
    sqrt((1-cos(theta))*(1+2*cos(theta))/(1+cos(theta))),0];
```

[23]:
```
A=[a1 a2 a3]
```

[23]:
```
4×3 Array{Float64,2}:
 1.0  0.838671  0.838671
 0.0  0.544639  0.248426
 0.0  0.0       0.484682
 0.0  0.0       0.0
```

[24]:
```
A'*A
```

[24]:
```
3×3 Array{Float64,2}:
 1.0       0.838671  0.838671
 0.838671  1.0       0.838671
 0.838671  0.838671  1.0
```

[ ]:
```
```

4(i) <u>Version 1</u> Find all diagonal square roots of I_7. How many are there?

$$\sqrt{I_7} = \begin{bmatrix} \pm 1 & & & & & & \\ & \pm 1 & & & & 0 & \\ & & \pm 1 & & & & \\ & & & \pm 1 & & & \\ & 0 & & & \pm 1 & & \\ & & & & & \pm 1 & \\ & & & & & & \pm 1 \end{bmatrix}$$

The diagonal square roots have either 1 or -1 on the diagonal. There are $2^7 = 128$ different ways to arrange the signs, so there are 128 diagonal square roots for I_7.

4. Find all diagonal square roots of I_6.
How many are there

Versions 2, 3, 8, 4

(i) $$\sqrt{I_6} = \begin{bmatrix} \pm 1 & & & & & \\ & \pm 1 & & & 0 & \\ & & \pm 1 & & & \\ & & & \pm 1 & & \\ & 0 & & & \pm 1 & \\ & & & & & \pm 1 \end{bmatrix}$$

Has either $+1$ or -1 on the diagonals.
There are 2^6 ways to do this, so I_6 has 64 diagonal square roots.

(ii) Find a non-diagonal matrix $A \in \mathbb{R}^{2\times 2}$ such that $A^2 = I$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Version 1

5. Consider the 6×6 matrix

(i) $A = \begin{bmatrix} I_5 & a \\ a^T & 0 \end{bmatrix}$ where $a \in \mathbb{R}^5$.

To be invertible we need a left or a right inverse. This happens when either the rows or the colums are linearly independent.

Clearly if $a = 0$ then A is not invertible. Suppose now that $a \neq 0$. If the columns were linearly dependent, this would mean there are $c_1, \ldots, c_6$ not all zero such that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ a_2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ a_3 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ a_4 \end{bmatrix} + c_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ a_5 \end{bmatrix} + c_6 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ 0 \end{bmatrix} = 0$$

Solving these equations for c_i's gives

$c_i = c_6 a_i$ for $i = 1, \dots, 5$
and $c_1 a_1 + \cdots + c_5 a_5 = 0$

Now if $c_6 = 0$ then $c_i = 0$ for all $i = 1, \dots, 5$ in which case all of the c_i's are zero.
Therefore $c_6 \neq 0$. Plugging into the second equation now yields that

$$c_1 a_1 + \cdots + c_5 a_5 = c_6 a_1^2 + \cdots c_6 a_5^2 = 0$$

or equivalently that

$$a_1^2 + \cdots + a_5^2 = 0.$$

However, since $a \neq 0$, this can't happen.
Therefore, the columns of A must be linearly independent.

In summary A is invertible if and only if $a \neq 0$.

<u>Versions 2,3,4.</u>

(i) answer is the same, as the size of the matrix wasn't used in the argument.

5(a) We search for the inverse one row at a time and look for a pattern.

$$\begin{bmatrix} I_5 & a \\ a^T & 0 \end{bmatrix} x = e_1$$

means

$$x_1 + a_1 x_6 = 1$$
$$x_2 + a_2 x_6 = 0$$
$$\vdots$$
$$x_5 + a_5 x_6 = 0$$

and $a_1 x_1 + \cdots + a_5 x_5 = 0$

Plug these in to get

$$a_1(1 - a_1 x_6) - a_2^2 x_6 - \cdots a_5^2 x_6 = 0$$

So that

$$x_6 \, a^T a = a_1 \quad \text{or} \quad x_6 = \frac{a_1}{a^T a}$$

and
$$x_1 = 1 - a_1 \frac{a_1}{a^T a}$$
$$x_2 = -a_2 \frac{a_1}{a^T a}$$
$$\vdots$$
$$x_5 = -a_5 \frac{a_1}{a^T a}$$

5(ii) continues...

Consequently

$$x = \begin{bmatrix} -a \\ \\ \\ 1 \end{bmatrix} \frac{a_1}{a^T a} + e_1$$

The same holds for $e_2, \dots, e_5$. So we have that

$$B = \left[\begin{array}{c|c} I_5 - \frac{a a^T}{a^T a} & ? \\ \hline \frac{a^T}{a^T a} & ? \end{array}\right]$$

To solve for the last column

$$\begin{bmatrix} I_5 & a \\ a^T & 0 \end{bmatrix} x = e_6$$

yields $\quad x_i + a_i x_6 = 0 \quad$ for $i = 1, \dots, 5$

and $\quad a_1 x_1 + \dots + a_5 x_5 = 1.$

5(ii) continues

Substituting obtains that

$$-a_1^2 x_6 - a_2^2 x_6 - \cdots - a_5^2 x_6 = 1$$

so $x_6 = -1/a^T a$

and $x_i = a_i(1/a^T a)$ for $i = 1, \ldots, 5$

Therefore

$$B = \left[\begin{array}{c|c} I_5 - \dfrac{a a^T}{a^T a} & \dfrac{a}{a^T a} \\ \hline \dfrac{a^T}{a^T a} & \dfrac{-1}{a^T a} \end{array}\right]$$

Now check that $B = A^{-1}$.

$$AB = \begin{bmatrix} I_5 & a \\ a^T & 0 \end{bmatrix} \begin{bmatrix} I_5 - \dfrac{a a^T}{a^T a} & \dfrac{a}{a^T a} \\ \dfrac{a^T}{a^T a} & \dfrac{-1}{a^T a} \end{bmatrix}$$

5(ii) continues...

$$= \left[\begin{array}{c|c} I_5 - \frac{aa^T}{a^Ta} + \frac{aa^T}{a^Ta} & \cancel{\frac{a}{a^Ta}} - \cancel{\frac{a}{a^Ta}} \\ \hline a^T - \frac{\cancel{a^Ta}\,a^T}{\cancel{a^Ta}} & \cancel{\frac{a^Ta}{a^Ta}} \end{array}\right]$$

$$= \begin{bmatrix} I_5 & 0 \\ 0 & 1 \end{bmatrix} = I_6$$

Therefore the inverse is

$$A^{-1} = \begin{bmatrix} I_5 - \frac{aa^T}{a^Ta} & \frac{a}{a^Ta} \\ \frac{a^T}{a^Ta} & \frac{-1}{a^Ta} \end{bmatrix}$$

The answers to the other versions are exactly the same except for the dimensions of the identity and the vector a.

q6v1

November 29, 2020

Question 6 version 1

[1]:
```
A=[8 -2 7 -2; -6 8 1 4; 1 8 -6 3; -1 -5 -9 2]
```

[1]:
```
4×4 Array{Int64,2}:
  8  -2   7  -2
 -6   8   1   4
  1   8  -6   3
 -1  -5  -9   2
```

[2]:
```
b=[8,5,-2,-1]
```

[2]:
```
4-element Array{Int64,1}:
  8
  5
 -2
 -1
```

[3]:
```
using LinearAlgebra
```

Part (i)

[6]:
```
Q,R=qr(A); Q=Matrix(Q);
```

[8]:
```
Q
```

[8]:
```
4×4 Array{Float64,2}:
 -0.792118   -0.174408   0.22006    0.541945
  0.594089   -0.436021   0.294804   0.608306
 -0.0990148  -0.741235  -0.653336  -0.117974
  0.0990148   0.479623  -0.661677   0.567752
```

[9]:
```
R
```

[9]:
```
4×4 Array{Float64,2}:
 -10.0995    5.04975  -5.24778   3.86158
   0.0     -11.4673   -1.52607  -2.65973
   0.0       0.0      11.7103   -2.54427
```

```
    0.0       0.0       0.0       2.13091
```

[10]:
```
Q*R
```

[10]:
```
4×4 Array{Float64,2}:
  8.0  -2.0   7.0  -2.0
 -6.0   8.0   1.0   4.0
  1.0   8.0  -6.0   3.0
 -1.0  -5.0  -9.0   2.0
```

Part (ii)

[11]:
```
y=Q'*b
```

[11]:
```
4-element Array{Float64,1}:
 -3.2674868918230247
 -2.57252224916199
  5.202848486712626
  7.045283976792092
```

Part (iii)

[15]:
```
x=R\y
```

[15]:
```
4-element Array{Float64,1}:
  0.6349480968858129
 -0.6972318339100344
  1.1626297577854667
  3.3062283737024214
```

[20]:
```
R*x
```

[20]:
```
4-element Array{Float64,1}:
 -3.2674868918230238
 -2.57252224916199
  5.202848486712624
  7.045283976792092
```

Part (iv)

[21]:
```
r=A*x-b
```

[21]:
```
4-element Array{Float64,1}:
 -4.440892098500626e-15
  0.0
  1.7763568394002505e-15
  8.881784197001252e-16
```

If we solved for x exactly, then r this would exactly equal zero. Since there is rounding error, then r should be close to zero. Note that, e-15 means that r is approximately zero to about 15 digits.

[ ]:
```

```

q6v2

November 29, 2020

Question 6 version 1

[22]:
```
A=[1 2 -1 1; -6 2 7 6; 7 -5 -7 7; -6 6 0 -8]
```

[22]:
```
4×4 Array{Int64,2}:
  1   2  -1   1
 -6   2   7   6
  7  -5  -7   7
 -6   6   0  -8
```

[25]:
```
b=[5,0,0,-4]
```

[25]:
```
4-element Array{Int64,1}:
  5
  0
  0
 -4
```

[26]:
```
using LinearAlgebra
```

Part (i)

[27]:
```
Q,R=qr(A); Q=Matrix(Q);
```

[28]:
```
Q
```

[28]:
```
4×4 Array{Float64,2}:
 -0.0905357   0.682806    0.544623  0.478502
  0.543214   -0.508428    0.157174  0.649396
 -0.63375    -0.0903405  -0.502587  0.581039
  0.543214    0.516832   -0.652755  0.108233
```

[29]:
```
R
```

[29]:
```
4×4 Array{Float64,2}:
 -11.0454  7.3334    8.32929  -5.61322
   0.0     3.90145  -3.60942  -7.1348
   0.0     0.0       4.0737    3.1916
```

```
   0.0      0.0        0.0        7.57629
```

[30]:
```
Q*R
```

[30]:
```
4×4 Array{Float64,2}:
  1.0   2.0  -1.0           1.0
 -6.0   2.0   7.0           6.0
  7.0  -5.0  -7.0           7.0
 -6.0   6.0  -7.27994e-16  -8.0
```

Part (ii)

[31]:
```
y=Q'*b
```

[31]:
```
4-element Array{Float64,1}:
 -2.6255366352330367
  1.3467040379564903
  5.334134151548588
  1.9595811961957295
```

Part (iii)

[32]:
```
x=R\y
```

[32]:
```
4-element Array{Float64,1}:
 2.163909774436089
 1.8421052631578934
 1.106766917293233
 0.2586466165413533
```

[33]:
```
R*x
```

[33]:
```
4-element Array{Float64,1}:
 -2.6255366352330345
  1.3467040379564903
  5.334134151548588
  1.9595811961957292
```

Part (iv)

[34]:
```
r=A*x-b
```

[34]:
```
4-element Array{Float64,1}:
 -3.552713678800501e-15
  5.773159728050814e-15
 -3.552713678800501e-15
  0.0
```

If we solved for x exactly, then r this would exactly equal zero. Since there is rounding error, then r should be close to zero. Note that, e-15 means that r is approximately zero to about 15 digits.

```
[ ]:
```

q6v3

November 29, 2020

Question 6 version 1

[35]:
```
A=[-7 1 -2 8; -8 4 9 -8; -6 5 5 4; -2 7 -3 -5]
```

[35]:
```
4×4 Array{Int64,2}:
 -7  1  -2   8
 -8  4   9  -8
 -6  5   5   4
 -2  7  -3  -5
```

[36]:
```
b=[-5,-8,-8,-9]
```

[36]:
```
4-element Array{Int64,1}:
 -5
 -8
 -8
 -9
```

[37]:
```
using LinearAlgebra
```

Part (i)

[41]:
```
Q,R=qr(A); Q=Matrix(Q);
```

[42]:
```
Q
```

[42]:
```
4×4 Array{Float64,2}:
 -0.565916   0.412569    0.71381   -1.96697e-16
 -0.646762   0.0501252  -0.541731  -0.534522
 -0.485071  -0.257374   -0.235812   0.801784
 -0.16169   -0.872372    0.376025  -0.267261
```

[43]:
```
R
```

[43]:
```
4×4 Array{Float64,2}:
 12.3693  -6.71015  -6.62931   -0.485071
  0.0     -6.7804    0.956235   6.23192
  0.0      0.0      -8.61034    7.22096
```

```
   0.0       0.0        0.0          8.81962
```

[44]:
```
Q*R
```

[44]:
```
4×4 Array{Float64,2}:
 -7.0  1.0  -2.0   8.0
 -8.0  4.0   9.0  -8.0
 -6.0  5.0   5.0   4.0
 -2.0  7.0  -3.0  -5.0
```

Part (ii)

[45]:
```
y=Q'*b
```

[45]:
```
4-element Array{Float64,1}:
 13.339459376998313
  7.446489658374844
 -0.7329302324912419
  0.26726124191242295
```

Part (iii)

[46]:
```
x=R\y
```

[46]:
```
4-element Array{Float64,1}:
  0.5666509656146964
 -1.054796671376982
  0.11053540587219356
  0.030303030303030148
```

[47]:
```
R*x
```

[47]:
```
4-element Array{Float64,1}:
 13.339459376998313
  7.446489658374844
 -0.7329302324912419
  0.26726124191242295
```

Part (iv)

[48]:
```
r=A*x-b
```

[48]:
```
4-element Array{Float64,1}:
 -3.552713678800501e-15
  1.7763568394002505e-15
  0.0
  0.0
```

If we solved for x exactly, then r this would exactly equal zero. Since there is rounding error, then r should be close to zero. Note that, e-15 means that r is approximately zero to about 15 digits.

[ ]:
```

```

q6v4

November 29, 2020

Question 6 version 1

[49]:
```
A=[5 -2 -5 -3; 9 -2 9 6; 5 -8 -2 -6; 9 9 0 -2]
```

[49]:
```
4×4 Array{Int64,2}:
 5  -2  -5  -3
 9  -2   9   6
 5  -8  -2  -6
 9   9   0  -2
```

[50]:
```
b=[0,2,-7,-4]
```

[50]:
```
4-element Array{Int64,1}:
  0
  2
 -7
 -4
```

[51]:
```
using LinearAlgebra
```

Part (i)

[52]:
```
Q,R=qr(A); Q=Matrix(Q);
```

[53]:
```
Q
```

[53]:
```
4×4 Array{Float64,2}:
 -0.343401  -0.186966  -0.601115  -0.69698
 -0.618123  -0.206847   0.713963  -0.255726
 -0.343401  -0.673305  -0.280445   0.59168
 -0.618123   0.684776  -0.224207   0.314226
```

[54]:
```
R
```

[54]:
```
4×4 Array{Float64,2}:
 -14.5602  -0.892844  -3.15929    0.618123
   0.0     12.3371     0.419812   1.99009
   0.0      0.0        9.99213    8.21821
```

```
0.0      0.0        0.0       -3.62195
```

[55]:
```
Q*R
```

[55]:
```
4×4 Array{Float64,2}:
 5.0  -2.0  -5.0          -3.0
 9.0  -2.0   9.0           6.0
 5.0  -8.0  -2.0          -6.0
 9.0   9.0  -5.04691e-15  -2.0
```

Part (ii)

[56]:
```
y=Q'*b
```

[56]:
```
4-element Array{Float64,1}:
  3.6400549446402586
  1.5603404591721186
  4.2878703286446935
 -5.910118924031819
```

Part (iii)

[57]:
```
x=R\y
```

[57]:
```
4-element Array{Float64,1}:
  0.023842485771419782
 -0.10567604983848661
 -0.912936471312106
  1.6317489616982008
```

[58]:
```
R*x
```

[58]:
```
4-element Array{Float64,1}:
  3.6400549446402586
  1.5603404591721186
  4.2878703286446935
 -5.910118924031819
```

Part (iv)

[59]:
```
r=A*x-b
```

[59]:
```
4-element Array{Float64,1}:
  0.0
  1.7763568394002505e-15
 -8.881784197001252e-16
 -2.6645352591003757e-15
```

If we solved for x exactly, then r this would exactly equal zero. Since there is rounding error, then r should be close to zero. Note that, e-15 means that r is approximately zero to about 15 digits.

[ ]:
```

```

#7 <u>version</u> 1&3 Let $A \in \mathbb{R}^{8\times 9}$ and define

$$A^\dagger = A^T(AA^T)^{-1}$$

Then

$$AA^\dagger A = A\left(A^T(AA^T)^{-1}\right)A$$

$$= \cancel{(AA^T)}\cancel{(AA^T)^{-1}}\,A = A$$

and

$$A^\dagger A A^\dagger = \left(A^T(AA^T)^{-1}\right)A\left(A^T(AA^T)^{-1}\right)$$

$$= A^T\cancel{(AA^T)^{-1}}\cancel{(AA^T)}(AA^T)^{-1}$$

$$= A^T(AA^T)^{-1} = A^\dagger$$

<u>Version</u> 2 Let $A \in \mathbb{R}^{6\times 8}$. The answer is the same because <u>rows<cols</u> and so the pseudo-inverse is defined in the same way and the checking doesn't depend on the exact dimensions.

#7 <u>version 4</u> Let $A \in \mathbb{R}^{9\times 7}$ and define the pseudo inverse

$$A^{\dagger} = (A^TA)^{-1}A^T$$

Then

$$AA^{\dagger}A = A\left((A^TA)^{-1}A^T\right)A$$

$$= A\,\cancel{(A^TA)^{-1}(A^TA)} = A$$

and

$$A^{\dagger}AA^{\dagger} = \left((A^TA)^{-1}A^T\right)A\left((A^TA)^{-1}A^T\right)$$

$$= (A^TA)^{-1}\cancel{(A^TA)(A^TA)^{-1}}A$$

$$= (A^TA)^{-1}A = A^{\dagger}.$$

8. Consider the 100×100 population dynamics matrix

$$A = \begin{bmatrix} b_1 & b_2 & \cdots & & b_{99} & b_{100} \\ 1-d_1 & 0 & \cdots & & 0 & 0 \\ & 1-d_2 & \ddots & & \vdots & \vdots \\ & & & \ddots & 0 & 0 \\ & 0 & & & 1-d_{99} & 0 \end{bmatrix}$$

where $b_i \geq 0$ and $0 \leq d_i \leq 1$ are the death rates. What are the conditions on which A is invertible.

A is invertible if it has a right inverse, which means that the rows must be linearly independent.

In particular none of the rows must be zero. This implies that

$$\boxed{d_i < 1 \quad \text{for all } i}$$

Note also that the columns must be linearly independent. In particular

8. continues—

The last column must be nonzero.
and consequently $\boxed{b_{100} > 0.}$

Further note that this guarantees the first row is independent from the remaining rows. Therefore these necessary conditions are also sufficient. In particular, for A to be invertible it is necessary and sufficient that

$$b_{100} > 0$$

and

$$d_i < 1 \text{ for all } i.$$