Singular value decomposition... Idea use the spectral theorem in the case where A is not symmetric...

let $B = A^T A$ where $A \in \mathbb{R}^{n \times n}$ arbitrary...

Add the simplifying assumption that A is invertible. to avoid dealing with a certain special case until later.

If A is invertible then B is invertible...

since $B^{-1} = A^{-1}(A^{-1})^T$ ... Check

$$BB^{-1} = A^T \cancel{A}\cancel{A^{-1}}(A^{-1})^T = A^T(A^{-1})^T$$

$$= (A^{-1}A)^T = I^T = I \,_{\prime\prime}$$

Since B is invertible all the eigenvalues of B are non-zero...

Since B is symmetric it has an orthonormal eigenbasis

Check $B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$

Therefore there are vectors x_i and value λ_i such that

$$B.x_i = \lambda_i x_i \quad , \quad x_i \cdot x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n$ ← there are enough eigenvectors to make a basis..

Relate the eigenvectors of B back to the matrix A. What happens is you do Ax_i?

Define $y_i = Ax_i$ for $i=1,\dots,n$ ... Claim the y_i vectors are orthogonal...

$$y_i \cdot y_j = Ax_i \cdot Ax_j = (Ax_i)^T Ax_j = x_i^T \underbrace{A^T A}_{B} x_j$$

$$= x_i^T B x_j = x_i \cdot Bx_j = x_i \cdot \lambda_j x_j$$

$$= \lambda_j x_i \cdot x_j = \begin{cases} \lambda_j & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Note: $\|y_i\| = \sqrt{\lambda_i}$

by the simplifying assumption That A is invertible none of the y's are zero because all the λ's are non-zero...

Therefore

$$z_i = \frac{y_i}{\sqrt{\lambda_i}} \text{ for } i=1,\dots,n \text{ is an orthonormal basis....}$$

Notation

$$U = \begin{bmatrix} x_1 \,\big|\, x_2 \,\big|\, \dots \,\big|\, x_n \end{bmatrix}, \quad V = \begin{bmatrix} z_1 \,\big|\, z_2 \,\big|\, \dots \,\big|\, z_n \end{bmatrix}$$

What do we know about U and V?

- V is square
- V has orthonormal columns

Therefore $V^T = V^{-1}$. Also $U^T = U^{-1}$.

Now...

$$AU = A\left[x_1 \middle| x_2 \middle| \cdots \middle| x_n\right] = \left[Ax_1 \middle| Ax_2 \middle| \cdots \middle| Ax_n\right] = \left[y_1 \middle| y_2 \middle| \cdots \middle| y_n\right]$$

also

$$V\begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \\ & & & \sqrt{\lambda_n} \end{bmatrix} = \left[z_1 \middle| z_2 \middle| \cdots \middle| z_2\right]\begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots & \sqrt{\lambda_n} \end{bmatrix}$$

$$= \left[\frac{y_1}{\sqrt{\lambda_1}} \middle| \frac{y_2}{\sqrt{\lambda_2}} \middle| \cdots \middle| \frac{y_n}{\sqrt{\lambda_n}}\right]\begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \\ & & & \sqrt{\lambda_n} \end{bmatrix} = \left[y_1 \middle| y_2 \middle| \cdots \middle| y_n\right]$$

Therefore

$$AU = V\Sigma \quad \text{where } \Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \\ & & & \sqrt{\lambda_n} \end{bmatrix}$$

or

$$A = V\Sigma U^T$$

The last factorization of a matrix in this course

orthogonal matrix

orthogonal matrix

diagonal matrix

This factorization combines the "best" things about the $A = QR$ factorization with the best things about the $A = SDS^{-1}$ factorization.

orthogonal matrix

upper triangular

using column ~~operation~~ and Gram-Schmidt algorithm...

invertible matrix

diagonal.