Last time we assumed A was invertible so the λ's are all non-zero. Then we defined

$$z_i = \frac{y_i}{\sqrt{\lambda_i}} \quad \text{for } i=1,\dots,n$$

and obtained an orthonormal basis.

If A is not invertible then some of the λ's are zero. which means it's not possible to define n vectors by renormalizing the y's.

Group the indices i where z_i can be defined together and the indices where $\lambda_i = 0$ in the other group.

Now, since the indices themselves were arbitrary, I can make the assumption

$$\lambda_i \neq 0 \quad \text{for } i=1,\dots,m$$

$$\lambda_i = 0 \quad \text{for } i=m+1,\dots,n$$

Traditionally we order the eigenvalues of B so that $\lambda_n \geq \lambda_{n-1} \geq \dots \geq \lambda_m \geq \lambda_{m-1} \geq \dots \geq \lambda_1$

Recall

the $\sqrt{\lambda_i}$ are called singular values of the matrix A.

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ 0 & & \ddots & \\ & & & \sqrt{\lambda_n} \end{bmatrix}. = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ 0 & & \ddots & \\ & & & \sigma_n \end{bmatrix}$$

The square roots of the eigenvalues of $B = A^TA$ are by definition the singular values of A.

Standard notation $\sigma_i = \sqrt{\lambda_i}$

singular values...

When A is not invertible some of the σ's are zero.

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_m & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

n-m zeros along the diagonal

also recall the v matrix from last time

$$V = \begin{bmatrix} z_1 & z_2 & \cdots & z_m \end{bmatrix}$$

ends at m since some Z's are missing.

is not not square

To make it square need to add some vectors

$$V = \left[\begin{array}{c|c|c|c|c} z_1 & z_2 & \cdots & z_m & \text{missing vectors} \end{array}\right]$$

Since V is supposed to be an orthogonal matrix the missing vectors need to be unit length and orthogonal no all the other vectors so in the end V has n orthonormal vectors as columns.

Use Gram-Schmidt process to find the missing vectors.

$$V = \left[\begin{array}{c|c|c|c|c|c} z_1 & z_2 & \cdots & z_m & z_{m+1} & \cdots\ z_n \end{array}\right]$$

just made these up so V is orthogonal.

Only thing left is to check that $A = V\Sigma U^T$ still...

Need to check AU is the same as $V\Sigma$.

Therefore $V^{-1} = V^T$. Also $U^{-1} = U^T$.

Now...

$$AU = A\left[\begin{array}{c|c|c|c} x_1 & x_2 & \cdots & x_n \end{array}\right] = \left[\begin{array}{c|c|c|c} Ax_1 & Ax_2 & \cdots & Ax_n \end{array}\right] = \left[\begin{array}{c|c|c|c} y_1 & y_2 & \cdots & y_n \end{array}\right]$$

also

$$= \left[\begin{array}{c|c|c|c|c|c} y_1 & \cdots & y_m & \overbrace{0 & & 0}^{n-m \text{ zero columns}} \end{array}\right]$$

$$V\Sigma = \begin{bmatrix} v_1 & \cdots & v_m & v_{m+1} & \cdots & v_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_m & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}$$

These are the same so we are done...

$$= \begin{bmatrix} \overbrace{\sqrt{\lambda_1}\, v_1 & \cdots & \sqrt{\lambda_m}\, v_m}^{y_i'} & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} y_1 & \cdots & y_m & 0 & \cdots & 0 \end{bmatrix}$$

since $v_i = \dfrac{y_i}{\sqrt{\lambda_i}}$

Now work an example from the book that we didn't have time for in class

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

First find the eigenvectors and eigenvalues of $B = A^TA$.

$$A^TA = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\det(B - \lambda I) = \det\begin{bmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = (8-\lambda)(5-\lambda) - 4 = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)$$

so the eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 4$

Note the eigenvalues are non-negative and we chose them in order so $\lambda_1 \geq \lambda_2$.

The corresponding eigenvectors:

$\lambda = 9$: $x \in \mathrm{Nul}\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$ so $-x_1 + 2x_2 = 0$ $\quad x_1 = 2x_2 \quad x_2 = \text{free}$

and $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2$ represent all the eigenvectors

To make a vector of $\underline{\underline{\text{unit}}}$ length normalize. Thus,

$$\lambda_1 = 9 \quad \text{and} \quad x_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}$$

$\underline{\underline{\lambda = 4}}$ $\quad x \in \text{Nul}\begin{bmatrix}4 & 2\\2 & 1\end{bmatrix}$ so $\quad 2x_1 + x_2 = 0 \qquad x_2 = -2x_1 \qquad x_1 = \text{free}$

and

$x = \begin{bmatrix}1\\-2\end{bmatrix}x_1$ represent all the eigenvectors

To make a vector of unit length normalize. Thus,

$$\lambda_2 = 4 \quad \text{and} \quad x_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}$$

In summary, B has an orthonormal basis of eigenvectors given by

$$\{x_1, x_2\} = \left\{\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}\right\}$$

so the matrix U is

By chance it turned out that U was symmetric

$$U = \begin{bmatrix}x_1 \,|\, x_2\end{bmatrix} = \begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\1/\sqrt{5} & -2/\sqrt{5}\end{bmatrix} \qquad \rightarrow \quad U^{-1} = U^T = \begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\1/\sqrt{5} & -2/\sqrt{5}\end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix}\sqrt{\lambda_1} & 0\\0 & \sqrt{\lambda_2}\end{bmatrix} = \begin{bmatrix}\sqrt{9} & 0\\0 & \sqrt{4}\end{bmatrix} = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$$

What's left is to find the matrix V. Recall $y_i = Ax_i$

$$y_1 = Ax_1 = \begin{bmatrix}2 & -1\\2 & 2\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}3\\6\end{bmatrix} = \frac{3}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$$

and

$$y_2 = Ax_2 = \begin{bmatrix}2 & -1\\2 & 2\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}4\\-2\end{bmatrix} = \frac{2}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}$$

Consequently, $\quad v_1 = \dfrac{y_1}{\sqrt{\lambda_1}} = \dfrac{\frac{3}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}}{3} = \dfrac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$ ← These should be unit vectors.

and $\quad v_2 = \dfrac{y_2}{\sqrt{\lambda_2}} = \dfrac{\frac{2}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}}{2} = \dfrac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}$ ↙ If they are not unit vectors then something went wrong.

It follows that

$$V = \left[v_1 \,\middle|\, v_2 \right] = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

and the singular value decomposition of A is

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = A = V \Sigma U^T = \underset{\substack{\nearrow \\ \text{orthogonal} \\ \text{matrix}}}{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}} \underset{\substack{\uparrow \\ \text{diagonal} \\ \text{matrix}}}{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}} \underset{\substack{\nwarrow \\ \text{orthogonal} \\ \text{matrix}}}{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}}$$

Will go over the above computation on Monday, do one more example and discuss the review sheet for the final exam. Please check for the review sheet over the weekend.