eigenvector–eigenvalue problem

Let $A \in \mathbb{R}^{n \times n}$ $\qquad Ax = \lambda x$

① Solved $\underbrace{\det(A - \lambda I)}$ $= 0$ for λ

This is a polynomial in λ of degree n. because one of the $n!$ terms which appear in the definition of determinant involve multiplying the elements on the diagonal of $A - \lambda I$ together.

The fundamental theorem of algebra states that a polynomial of degree n has n roots (counted by multiplicity). Also note if the matrix A is chosen randomly the chances of a repeated root is statistically 0. Matrices that come from applications are not random and may lead to repeated roots.

→ Simplest case... there are n eigenvalues all of them are different.

label them $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$

In the example

et $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, u $\qquad \lambda_1 = 7 \qquad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -4 \qquad x_2 = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$

for each eigenvalue there is an eigenvector.

Thus

$$Ax_1 = \lambda_1 x_1 , \quad Ax_2 = \lambda_2 x_2 , \ldots , \quad Ax_n = \lambda_n x_n$$

Observation: the eigenvectors corresponding to different eigenvalues are linearly independent.

~~By contradiction, suppose the x_i's were dependent.~~

~~then $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$~~

~~with not all of the $c_i = 0$. Suppose $c_k \neq 0$.~~

~~Now use the fact that the x_i's are eigenvectors...~~

~~$$A(c_1x_1 + c_2x_2 + \cdots + c_nx_n) = A0$$~~

~~$$c_1Ax_1 + c_2Ax_2 + \cdots + c_nAx_n = 0$$~~

~~$$c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n = 0$$~~

By contradiction, suppose the x_i's were dependent.

Choose p so that $x_1, \ldots, x_p$ are independent
but $x_1, \ldots, x_{p+1}$ are dependent.

Thus

$$x_{p+1} = c_1x_1 + c_2x_2 + \cdots + c_px_p$$

and so

$$A\, x_{p+1} = A(c_1x_1 + c_2x_2 + \cdots + c_px_p)$$

$$\lambda_{p+1} x_{p+1} = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_p \lambda_p x_p$$

$$\lambda_{p+1} x_{p+1} = \lambda_{p+1}\left(c_1 x_1 + c_2 x_2 + \cdots + c_p x_p\right)$$

and so

Subtract

$$\lambda_{p+1} x_{p+1} = c_1 \lambda_{p+1} x_1 + c_2 \lambda_{p+1} x_2 + \cdots + c_p \lambda_{p+1} x_p$$

$$\lambda_{p+1} x_{p+1} = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_p \lambda_p x_p$$

$$0 = c_1(\lambda_{p+1} - \lambda_1)x_1 + c_2(\lambda_{p+1} - \lambda_2)x_2 + \cdots + c_p(\lambda_{p+1} - \lambda_p)x_p$$

Since the λ_i's are all different then

$\lambda_{p+1} - \lambda_i \neq 0$ for $i = 1, \ldots, p$

Therefore not all $c_i(\lambda_{p+1} - \lambda_i) = 0$ so that implies $x_1, \ldots, x_p$ are dependent… which is a contradiction… therefore

$\boxed{\text{the eigenvectors corresponding to different eigenvalues are linearly independent.}}$