Recall: $Ax = \lambda x$ solve for x and λ.

$$(A - \lambda I)x = 0$$

Used determinants to deduce that if $x \neq 0$

then $\mathrm{Nul}(A - \lambda I)$ is non-trivial

so $A - \lambda I$ has free variables

so $\underbrace{\det(A - \lambda I)}$ $= 0$

characteristic polynomial... $\chi(\lambda)$...

What is the degree of this polynomial?

$A \in \mathbb{R}^{n \times n}$

Example $n = 4$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 0 \\ 6 & 1 & 6 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 & 4 \\ 1 & 1-\lambda & 1 & 1 \\ 2 & 3 & 1-\lambda & 0 \\ 6 & 1 & 6 & 3-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 3 & 4 \\ 1 & 1-\lambda & 1 & 1 \\ 2 & 3 & 1-\lambda & 0 \\ 6 & 1 & 6 & 3-\lambda \end{bmatrix}\right)$$

TION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

The definition of determinant essentially adds up all the different products obtained by choosing a different column for each row while changing the ± back and forth. Since there are n rows and n columns, this is another way of seeing there are $n!$ things that needed to be added up... One of those terms is the product on the diagonal, thus

$4! = 4\cdot3\cdot2\cdot1$

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & 3 & 4 \\ 1 & 1-\lambda & 1 & 1 \\ 2 & 3 & 1-\lambda & 0 \\ 6 & 1 & 6 & 3-\lambda \end{bmatrix}\right) \approx \overset{\text{plus}}{(+)} \underbrace{(1-\lambda)(1-\lambda)(1-\lambda)(3-\lambda)}_{\text{from this term you already see its a poly of degree 4.}} + \text{23 other terms}$$

If $A \in \mathbb{R}^{n \times n}$ then $\det(A - \lambda I)$ is a polynomial of degree n. By the fundamental theorem of algebra the are n roots counted by multiplicity to the equation $\det(A - \lambda I) = 0$.

These values of λ may be complex, real or imaginary and may be repeated...

If the roots λ are all different, then there are n eigenvectors one for each distinct value of λ. Thus

$$Ax_1 = \lambda_1 x_1, \quad Ax_2 = \lambda_2 x_2 \quad \ldots \quad Ax_n = \lambda_n x_n$$

Result from last time...

The eigenvectors corresponding to different eigenvalues are linearly independent.

If the λ's are chosen randomly, they'll all be different, in which case

$$\underbrace{x_1, x_2, \ldots, x_n}_{n \text{ of them}} \quad \text{are all lin. indept}$$

This means $\underbrace{\{x_1, x_2, \ldots, x_n\}}_{\text{eigenbasis}}$ form a basis of $\mathbb{R}^n$

Note: (1) From a statistical point of view this always happens if A was chosen randomley.

(2) A is not usually chosen randomly.

(3) If $A^T = A$ then there is an eigenbasis for all such A.

In application if the original matrix is not symmetric in the sense of (3) above, sometimes one can use a symmetric matrix in it's place.

What can go wrong? If there is a repeated root to $\det(A - \lambda I) = 0$, you may or may not have an eigenbasis

An eigenbasis is important because then any vector $x \in \mathbb{R}^n$ can be written as

$$x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

and so

$$Ax = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots + c_n \lambda_n x_n$$

can be used to turn the Matrix-vector multiplication Ax into (a sum of) scalar multiplications, which simplifies things...

Example $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 = 0$$

repeated root $\lambda = 2$ with mult 2.

Find all eigenvectors for $\lambda = 2$ by computing

$$\text{Nul}(A - 2I) = \text{Nul}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} : x_1 \neq 0 \right\}$$

one dim. space

$$= \left\{ x : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \right\}$$

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$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$ $\quad 0 \cdot x_1 + 1 \cdot x_2 = 0$ $\quad$ thus $x_2 = 0$

$\quad 0 \cdot x_1 + 0 \cdot x_2 = 0$ $\quad x_1 = \text{free}$

$\boxed{x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1}$ ← all solutions...

only 1 free variable means only one linearly independent eigenvector for $\lambda = 2$.