Chapter 5.2

11. $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$

Solve $Ax = \lambda x$ when $A =$

Method is to use determinants and the characteristic polynomial to solve for the λ's first and then plug it in to find the corresponding x's.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} 4-\lambda & 0 & 0 \\ 5 & 3-\lambda & 2 \\ -2 & 0 & 2-\lambda \end{bmatrix}$$

expand along this row

$$\det A_{11} = \det \begin{bmatrix} 3-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda) - 0 \cdot 2$$

$$= (-1)^{1+1}(4-\lambda)\det A_{11} - 0 + 0$$

$$= (4-\lambda)(3-\lambda)(2-\lambda) = 0$$

Thus $\lambda = 2, 3$ or 4

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Generalized:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \text{for any fixed } i$$

Since $\det A^T = \det A$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \text{for any fixed } j$$

Recall... **11.** $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$ and $\lambda = 2, 3$ or 4

Plug λ in and solve $\mathrm{Nul}(A - \lambda I)$ each time.

$\lambda = 2$: $(A - 2I)x = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix} x = 0$

make reduced row eschelon form:

$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix}$

$r_2 \leftarrow r_2 - \frac{5}{2} r_1$

$r_3 \leftarrow r_3 + r_1$

$r_1 \leftarrow \frac{1}{2} r_1$

$$\begin{array}{ccc} P & P & F \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$1x_2 + 2x_3 = 0$

$x_1 = 0$
$x_2 = -2x_3$

$$x = \begin{bmatrix} 0 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} x_3$$

eigenvector...

$\lambda = 2$ and $x = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ is one eigenvalue eigenvector pair

$\lambda = 3$

$$A - 3I = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & 2 \\ -2 & 0 & -1 \end{bmatrix}$$

$r_2 \leftarrow r_2 - 5r_1$
$r_3 \leftarrow r_3 + 2r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$r_3 \leftarrow r_3 + \frac{1}{2} r_2$
$r_2 \leftarrow \frac{1}{2} r_2$

reduced row echelon form

$$\begin{array}{ccc} P & F & P \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_3 = 0$
x_2 free

$$x = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2$$

eigenvector

$\lambda = 3$ and $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ another eigenvalue eigenvector pair...

$\lambda = 4$

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 5 & -1 & 2 \\ -2 & 0 & -2 \end{bmatrix}$$

$r_3 \longleftrightarrow r_1$

$$\begin{bmatrix} -2 & 0 & -2 \\ 5 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_2 \leftarrow r_2 + \frac{5}{2} r_1$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} r_1 &\leftarrow \frac{-1}{2} r_1 \\ r_2 &\leftarrow -r_2 \end{aligned}$$

$$\begin{array}{ccc} P & P & F \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_3 = 0 \qquad x_1 = -x_3$$

$$x_2 + 3x_3 = 0 \qquad x_2 = -3x_3$$

$$x = \begin{bmatrix} -x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} x_3$$

eigenvector

$$\lambda = 4 \quad \text{and} \quad x = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

one more eigenvalue eigenvector pair