During this lecture there were difficulties with the classroom projector. I have added this green commentary to help anyone who might have missed something...

Goal solve $Ax = \lambda x$ by using determinats to find λ and the plugging the λ in to find x.

Eigenvalue-eigenvector example:

12. $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

3,3 minor

$$\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

expand on the row

3,3 entry

Expand determinant along a row

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \text{for some } i \text{ fixed}$$

take $i=3$ in this case

alternatively $\det A = \det A^T$

there are two natural choices to minimize number of terms

or a column

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \text{for some } j \text{ fixed.}$$

take $j=2$ for the second column...

Do the row:

$$\det(A-\lambda I) = 0 - 0 + (-1)^{3+3}(2-\lambda)\det\begin{bmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left\{(-1-\lambda)(4-\lambda) - 0(-3)\right\}$$

$$= (2-\lambda)(-1-\lambda)(4-\lambda) = 0$$

Therefore $\lambda=2$, $\lambda=-1$ or $\lambda=4$

$$A=\begin{bmatrix}-1 & 0 & 1\\ -3 & 4 & 1\\ 0 & 0 & 2\end{bmatrix}$$

12.

recall we're solving $Ax=\lambda x$

or equivalently $(A-\lambda I)x=0$

Now substitute to find x

reduced row eschelon form

$\lambda=2$

$$A-2I=\begin{bmatrix}-3 & 0 & 1\\ -3 & 2 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$r_2 \leftarrow r_2 - r_1$

$$\begin{bmatrix}-3 & 0 & 1\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$r_1 \leftarrow -\frac{1}{3}r_1$

$r_2 \leftarrow \frac{1}{2}r_2$

$$\begin{array}{ccc} P & P & F \end{array}$$

$$\begin{bmatrix}1 & 0 & -1/3\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

reduced row eschelon form

$x_1 - \frac{1}{3}x_3 = 0$

$x_2 = 0$

$$\begin{cases} x_1 = \frac{1}{3}x_3 \\ x_2 = 0 \\ x_3 \text{ free}\end{cases}$$

$$x=\begin{bmatrix}1/3\,x_3\\ 0\\ x_3\end{bmatrix}=\begin{bmatrix}1/3\\ 0\\ 1\end{bmatrix}x_3$$

eigenvector

Note

$$\mathrm{Nul}(A-2I)=\left\{\begin{bmatrix}1/3\\ 0\\ 1\end{bmatrix}c : c\in\mathbb{R}\right\}$$

any non-zero vector in this nullspace is an eigenvector for the eigenvalue $\lambda=2$

another choice is $\begin{bmatrix}1\\ 0\\ 3\end{bmatrix}$

$\lambda=2$ and $x=\begin{bmatrix}1\\ 0\\ 3\end{bmatrix}$ is one eigenvalue eigenvector pair...

$\lambda = -1$

$$A - (-1)I = \begin{bmatrix} 0 & 0 & 1 \\ -3 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad r_2 \leftrightarrow r_1$$

$$\begin{array}{ccc} P & F & P \end{array}$$
$$\begin{bmatrix} -3 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{matrix} r_3 \leftarrow r_3 - 3r_2 \\ r_1 \leftarrow r_1 - r_2 \end{matrix}$$

$$\begin{bmatrix} -3 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow -\frac{1}{3} r_1$$

reduced row eschelon form

$$\begin{array}{ccc} P & F & P \end{array}$$
$$\begin{bmatrix} 1 & -5/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \underbrace{x_1 - \frac{5}{3}x_2 = 0}_{x_1 = \frac{5}{3}x_2} \\ x_2 \text{ free} \\ x_3 = 0 \end{matrix}$$

$$x = \begin{bmatrix} 5/3\, x_2 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1 \\ 0 \end{bmatrix} x_2$$

eigenvector...

$\lambda = -1$ and $x = \begin{bmatrix} 5/3 \\ 1 \\ 0 \end{bmatrix}$ is another eigenvalue eigenvector pair.

The last eigenvalue was done after class. The method is the same as the previous two and included for completeness of the problem.

$\lambda = 4$

12. $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Thus $A - 4I = \begin{bmatrix} -5 & 0 & 1 \\ -3 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ $\quad r_2 \leftarrow r_2 - \frac{3}{5} r_1$

$\begin{bmatrix} -5 & 0 & 1 \\ 0 & 0 & 2/5 \\ 0 & 0 & -2 \end{bmatrix}$ $\quad r_3 \leftarrow r_3 + 5 r_2$, $\quad r_1 \leftarrow r_1 - \frac{5}{2} r_2$

$\begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 2/5 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad r_1 \leftarrow -\frac{1}{5} r_1$, $\quad r_3 \leftarrow \frac{5}{2} r_3$

reduced row echelon form

$$\begin{array}{c} \begin{array}{ccc} P & F & P \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x_1 = 0$
$x_2 = \text{free}$
$x_3 = 0$

$x = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2$

eigenvector

$\lambda = 4$ and $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the final eigenvalue eigenvector pair.