- The diagonalization of a matrix $A \in \mathbb{R}^{n \times n}$ by means of an eigenbasis... similarity transformation by S
- The factorization of: $A = SDS^{-1}$ where D is a diagonal matrix and S is the matrix whose columns are the eigenvectors of A.

note S is invertible... and this is the same as A having an eigenbasis...

From Wednesday...

11. $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$

1. $\lambda_1 = 2$ and $x_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$
2. $\lambda_2 = 3$ and $x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
3. $\lambda_3 = 4$ and $x_3 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$ $\boxed{\checkmark}$

$$\begin{array}{r} -5 \\ -9 \\ 2 \\ \hline -12 \end{array} \qquad \begin{array}{r} 2 \\ 0 \\ 2 \\ \hline 4 \end{array}$$

Check answers... plug them in to $Ax = \lambda x$.

Idea... everyone check one of the above answers

$$Ax = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -12 \\ 4 \end{bmatrix} \text{ and } \lambda x = 4 \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -12 \\ 4 \end{bmatrix}$$

the same

Therefore we have 3 eigenvectors corresponding to 3 different eigenvalues...

By the theorem $\boxed{}$ we know the eigenvectors are linearly independent. Therefore

$$\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$$

is a eigenbasis...

$$S = \begin{bmatrix} 0 & 0 & -1 \\ -2 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$$

Now multiply A by S

$$AS = A \begin{bmatrix} 0 & 0 & -1 \\ -2 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix} = A \left[x_1 \middle| x_2 \middle| x_3 \right] = \left[Ax_1 \middle| Ax_2 \middle| Ax_3 \right]$$

since the x_i's are eigenvectors

$$= \left[\lambda_1 x_1 \middle| \lambda_2 x_2 \middle| \lambda_3 x_3 \right] = \underbrace{\left[x_1 \middle| x_2 \middle| x_3 \right]}_{S} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{D}$$

Column operations matrix mult. on the right
$\left\{ \begin{array}{l} C_1 \leftarrow \lambda_1 C_1 \\ C_2 \leftarrow \lambda_2 C_2 \\ C_3 \leftarrow \lambda_3 C_3 \end{array} \right.$

Therefore...

$$AS = SD$$

$$ASS^{-1} = SDS^{-1}$$

$$\boxed{A = SDS^{-1}}$$

What can be done with the factorization $A = SDS^{-1}$

$$A^2 = AA = (SDS^{-1})(SDS^{-1}) = SD(S^{-1}S)DS^{-1} = SD^2S^{-1}$$

So squaring A is the same as squaring D inside the similarity...

Note

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

Thus

$$D^k = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}^k = A^k = SD^kS^{-1} = S\begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}S^{-1}$$

This is a computation savings (as well as a simplification) when k is really big...

Let $p(t) = 3t^2 + t + 5$. What is $p(A)$?

composition of a polynomial function with a linear function

$$p(A) = 3A^2 + A + 5I =$$

use eigenvector eigenvalue problem to simplify this