Let $p(t) = 3t^2 + t + 5$. What is $p(A)$?

$p(A) = 3A^2 + A + 5I =$

composition of a polynomial function with a linear function

use eigenvector eigenvalue problem to simplify this

Assume $A \in \mathbb{R}^{n \times n}$ has an eigenbasis $\{x_1, x_2, \dots, x_n\}$

Thus setting $S = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$

$$AS = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

where λ's are the eigenvalues corresponding to the eigenvectors.

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} = SD$$

Therefore $A = SDS^{-1}$ or equivalently $D = S^{-1}AS$.

$p(A) = 3A^2 + A + 5I = 3(SDS^{-1})^2 + SDS^{-1} + 5I$

recall $(SDS^{-1})^2 = SD\underbrace{S^{-1}S}_{SS^{-1}}DS^{-1} = SD^2S^{-1}$

$p(A) = 3SD^2S^{-1} + SDS^{-1} + 5I$

$$= S(3D^2S^{-1} + DS^{-1} + 5S^{-1}) = S(\underbrace{3D^2 + D + 5I}_{p(D)})S^{-1}$$

Therefore

$$P(A) = S\, P(D)\, S^{-1}$$

or in general for any polynomial and any matrix B and any invertible matrix S.

$$\boxed{P(SBS^{-1}) = S\,P(B)\,S^{-1}}$$

This trick works for more than just polynomials. It also work for power series, that is, infinitely long polynomials...

Recall the series expansions of e^x, $\sin x$ and $\cos x$.

note, since $e^{i\theta} = \cos\theta + i\sin\theta$ there is really only one power series to remember for these 3 functions...

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

notational convention

$$= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$ ← compact way to write the series expansion of e^x.

Note $i^0, i^1, i^2, i^3, i^4, i^5, i^6, \dots$ is the same as

every other term is negative the previous...

$1, i, -1, -i, 1, i, -1, \dots$

every other one is negative the previous odd powers of x in the e^x series with alternating signs

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \dots$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots$$

Note the radius of convergence for these power series is infinite. That is, they converge for any value of x...

Idea... compose sin with the linear function represented by the matrix A.

$$\sin A = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \frac{1}{7!}A^7 + \frac{1}{9!}A^9 - \dots$$

Since these power series converge for any x then they also converge for any matrix $A \in \mathbb{R}^{n\times n}$.

Assume A has an eigenbasis, then

$$A = SDS^{-1}$$ and

$$\sin A = \sin(SDS^{-1}) = S\sin(D)S^{-1}$$

and because D is diagonal

$$\sin(D) = \sin\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & \ddots & \lambda_n\end{bmatrix} = \begin{bmatrix}\sin\lambda_1 & & 0\\ & \sin\lambda_2 & \\ 0 & \ddots & \sin\lambda_n\end{bmatrix}$$