Last time...

Matrix $A \in \mathbb{R}^{n\times n}$ and eigenbasis $\{x_1, x_2, \ldots, x_n\}$

Set $S = \left[x_1 \middle| x_2 \middle| \ldots \middle| x_n \right]$ Then $A = SDS^{-1}$ where

D is diagonal $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

Recall the 3x3 matrix from before...

This is not squaring a table of numbers

It's composition of a linear function with itself...

$$\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}^2 \neq \begin{bmatrix} (4)^2 & 0^2 & 0^2 \\ (5)^2 & (3)^2 & (2)^2 \\ (-2)^2 & 0^2 & (2)^2 \end{bmatrix}$$

What is this? This is treating things as just a table of numbers...

$A = SDS^{-1}$ where

$$D^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}^2 = \begin{bmatrix} 2^2 & 0^2 & 0^2 \\ 0^2 & 3^2 & 0^2 \\ 0^2 & 0^2 & 4^2 \end{bmatrix}$$

actually works, just by chance or maybe design.

This comes from the dot product $\begin{bmatrix}2\\0\\0\end{bmatrix}\cdot\begin{bmatrix}0\\3\\0\end{bmatrix}=0=0^2$ just lucky...

$$D^2=\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}=\begin{bmatrix}2^2&\boxed{0}&\\&&\\&&\end{bmatrix}$$

Recall ...

$$\sin x = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots$$

$$\sin D = D-\frac{1}{3!}D^3+\frac{1}{5!}D^5-\frac{1}{7!}D^7$$

$$=\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}-\frac{1}{3!}\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}^3+\frac{1}{5!}\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}^5-\frac{1}{7!}\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}^7+\cdots$$

$$=\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}-\frac{1}{3!}\begin{bmatrix}2^3&0&0\\0&3^3&0\\0&0&4^3\end{bmatrix}+\frac{1}{5!}\begin{bmatrix}2^5&0&0\\0&3^5&0\\0&0&4^5\end{bmatrix}-\frac{1}{7!}\begin{bmatrix}2^7&0&0\\0&3^7&0\\0&0&4^7\end{bmatrix}+\cdots$$

$$=\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}-\begin{bmatrix}\frac{2^3}{3!}&0&0\\0&\frac{3^3}{3!}&0\\0&0&\frac{4^3}{3!}\end{bmatrix}+\begin{bmatrix}\frac{2^5}{5!}&0&0\\0&\frac{3^5}{5!}&0\\0&0&\frac{4^5}{5!}\end{bmatrix}-\begin{bmatrix}\frac{2^7}{7!}&0&0\\0&\frac{3^7}{7!}&0\\0&0&\frac{4^7}{7!}\end{bmatrix}+\cdots$$

$$=\begin{bmatrix}2-\frac{1}{3!}2^3+\frac{1}{5!}2^5-\frac{1}{7!}2^7+\cdots&0&0\\0&3-\frac{1}{3!}3^3+\frac{1}{5!}3^5-\frac{1}{7!}3^7+\cdots&0\\0&0&4-\frac{1}{3!}4^3+\frac{1}{5!}4^5-\frac{1}{7!}4^7+\cdots\end{bmatrix}$$

$$=\begin{bmatrix}\sin 2&0&0\\0&\sin 3&0\\0&0&\sin 4\end{bmatrix}$$

Note all this requires the matrix A has an eigenbasis... and we know some matrices don't. For example,

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Amazing result... Every symmetric matrix has an orthonormal eigenbasis...

Note in applications many matrices are naturally symmetric, such as a matrix used to represent a derivative. If they aren't symmetric, it's often possible to create a related matrix that is...

The advantage is not only guaranteed eigenbasis but an orthonormal one...

$$S = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$ ← orthogonal matrix

orthonormal vectors

Thus

$$S^{-1} = S^T.$$