$$A x \cdot \bar{x} = \lambda x \cdot \bar{x} = \lambda \|x\|^2$$

Since A real then $\bar{A} = A$

$$\| $$

Symmetry of A

$$(Ax)^T \bar{x} = x^T A^T \bar{x} = x^T A \bar{x} = x^T \bar{A} \bar{x}$$

Since A is real

$$= x^T \overline{Ax} = x^T \overline{\lambda x} = \bar{\lambda} x^T \bar{x} = \bar{\lambda} \|x\|^2$$

- If $\boxed{A \in \mathbb{R}^{n \times n} \text{ and } A^T = A}$ then the eigenvalues of A are real.

Note, this also means the eigenvectors are real.

same hypothesis....

Spectral Theorem: There is an orthonormal eigenbasis

- If λ_1 and λ_2 are two different eigenvalues with corresponding eigenvectors x_1 and x_2 then $x_1 \cdot x_2 = 0$.

We already know that λ_1 and λ_2 are real from the previous result. Also x_1 and x_2 are real...

Recall eigenvalues and eigenvectors mean

$$A x_1 = \lambda_1 x_1 \quad \text{and} \quad A x_2 = \lambda_2 x_2$$

symmetry

$$A x_1 \cdot x_2 = (Ax_1)^T x_2 = x_1^T A^T x_2 = x_1^T A x_2 = x_1 \cdot A x_2$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\lambda_1 x_1 \cdot x_2 \qquad\qquad\qquad\qquad x_1 \cdot \lambda_2 x_2$$

Put on other side

Therefore $\lambda_1 x_1 \cdot x_2 = \lambda_2 x_1 \cdot x_2$

Solve for $x_1 \cdot x_2$!

$$\lambda_1 x_1 \cdot x_2 - \lambda_2 x_1 \cdot x_2 = 0$$

factor out

$$(\lambda_1 - \lambda_2)\, x_1 \cdot x_2 = 0$$

Note since $\lambda_1 \neq \lambda_2$
then $\lambda_1 - \lambda_2 \neq 0$
and we can divide

thus $x_1 \cdot x_2 = 0$

Note by normalizing the eigenvectors as

$$\frac{x_1}{\|x_1\|} \quad \text{and} \quad \frac{x_2}{\|x_2\|}$$

I now have orthonormal vectors...

Two things left: To show

1. $A^T = A$ implies There is a basis.
2. What to do when an eigenvalue is repeated

Suppose x_1 and x_2 are linearly independent eigenvectors that have the same eigenvalue... Use Gram-Schmidt algorithm.

$$t_1 = x_1 \qquad q_1 = \frac{t_1}{\|t_1\|}$$

$$t_2 = x_2 - (q_1 \cdot x_2)\, q_1 \qquad q_2 = \frac{t_2}{\|t_2\|}$$

note if there were three linearly independent eigenvector all with the same eigenvalue, use Gram-schmidt on all three of them ...

The vectors q_1 and q_2 are also eigenvector and they are now orthonormal...

$$\left[\, x_1 \,\middle|\, x_1 \,\right] = \left[\, q_1 \,\middle|\, q_2 \,\right] \begin{bmatrix} \|t_1\| & (q_1 \cdot x_2) \\ 0 & \|t_2\| \end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

mult both sides on the right by R^{-1}

Fact

$$[q_1 | q_2] = [x_1 | x_2] R^{-1}$$

$$A [q_1 | q_2] = [Aq_1 | Aq_2]$$

$$\|$$

$$A [x_1 | x_2] R^{-1} = [Ax_1 | Ax_2] R^{-1} = [\lambda x_1 | \lambda x_2] R^{-1}$$

since the same λ in both columns, factor it out

$$[Aq_1 | Aq_2] = \lambda [x_1 | x_2] R^{-1} = \lambda [q_1 | q_2]$$

Therefore

$$[Aq_1 | Aq_2] = [\lambda q_1 | \lambda q_2]$$

So q_1 and q_2 are eigenvectors with eigenvalue λ again...