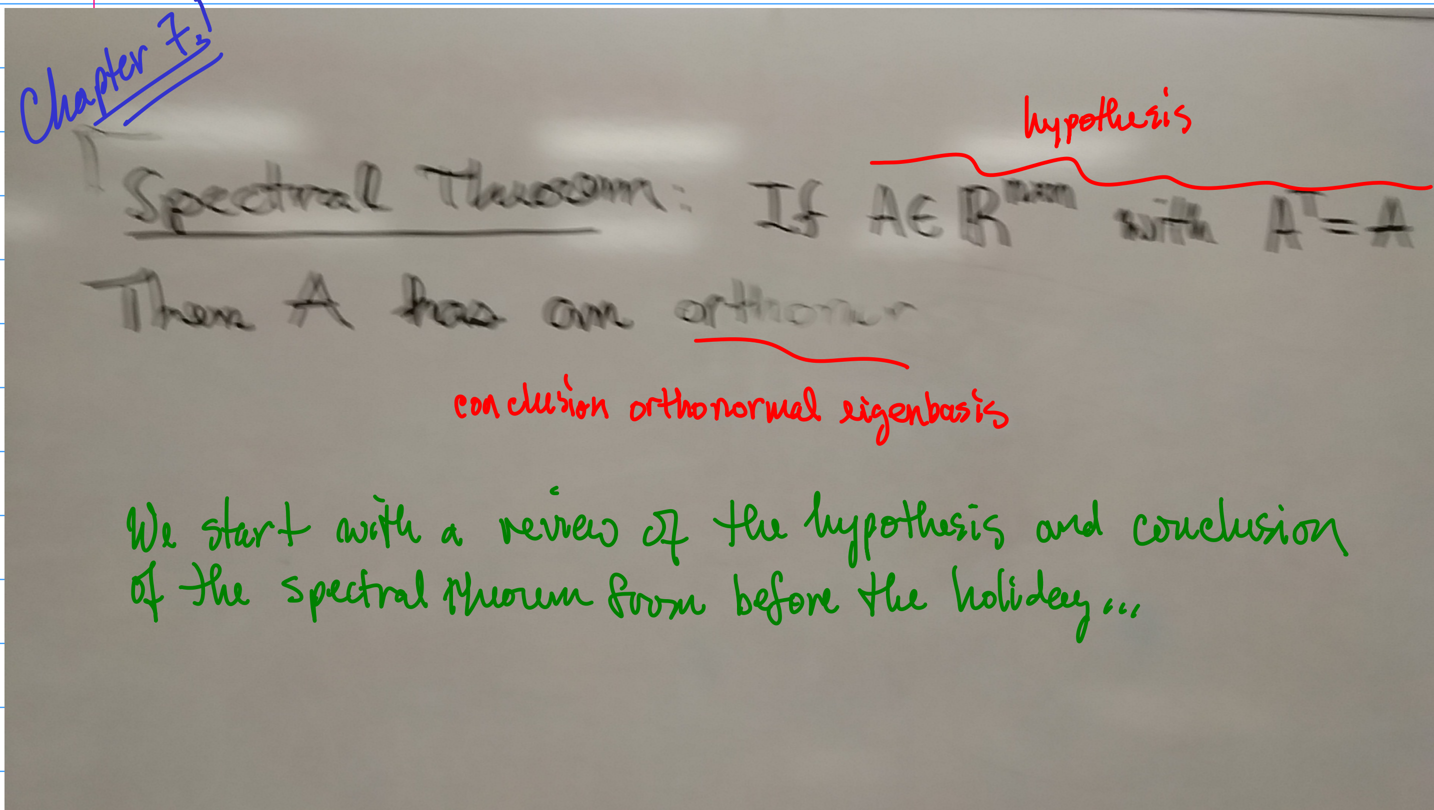
The lecture was disrupted by the projector not working and then by the projector repair people... Here are photos of the whiteboard that I made with my mobil phone.. It seems some are blurry. I'll fill in the details here as I post things...



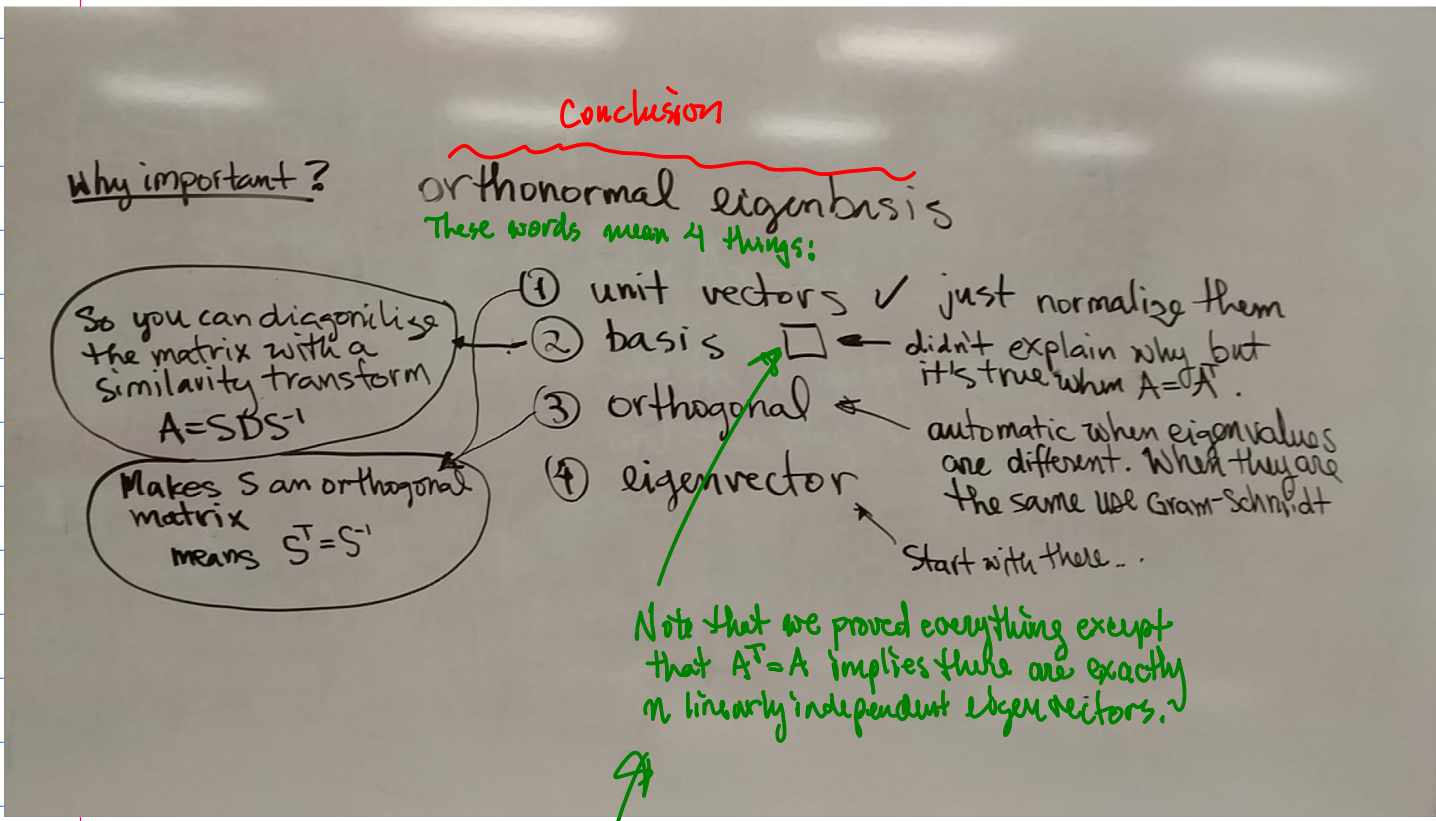
We start with a review of the hypothesis and conclusion of the spectral theorem from before the holiday...

The hypothesis $A \in \mathbb{R}^{n \times n}$ and $A^T = A$ are

1. easy to check
2. Occur naturally in some applications
3. When $A \neq A^T$ sometime a related matrix that is symmetric can be used instead.

That's the idea of the Singular Value Decomposition that is the focus of this last week...

What's the conclusion of the spectral theorem and why is it useful...



This result can be obtained by an induction argument based on a reduction that works because A is symmetric. It's not difficult, but not the focus of this class, so we skip it.. Instead we move to applications of the Spectral Theorem...

Two things left to do:

① Work a numerical example where $A = SDS^T$ to see how things work...

② Discuss the Singular Value Decomposition, which is an application of the spectral theorem to matrices which are not symmetric.

Since the projector is broken we start with the singular value decomposition, as that involves less writing...

Sorry, this slide turned out blurry too...

Chapter 7.4

Singular Value decomposition

Idea to make a symmetric matrix out of one that is not symmetric.

Given $A \in \mathbb{R}^{m\times n}$ but not symmetric. Define $B = A^TA$

Claim $B = B^T$. Why $B^T = (A^TA)^T = A^T A^{TT} = A^TA = B$

By the spectral Theorem B has an orthonormal eigenbasis.

$Bx_i = \lambda_i x_i$ and $x_i \cdot x_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

for $i = 1, 2, \dots, n$

Thus $B = SDS^T$ where $S = \left[x_1 \middle| x_2 \middle| \cdots \middle| x_n \right]$ and $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$

Since B is symmetric the spectral theorem implies that S is an orthogonal matrix. Thus $S^{-1} = S^T$.

orthonormal columns

diagonal matrix

$B = A^TA$, $Bx_i = \lambda_i x_i$ Let $y_i = Ax_i$ Claim y_i are orthogonal

eigenvectors of B

not necessarily unit vectors

<u>Claim</u> $y_i \cdot y_j = 0$ if $i \neq j$

$$y_i \cdot y_j = Ax_i \cdot Ax_j = (Ax_i)^T A x_j = x_i^T \overbrace{A^TA}^{B} x_j = x_i \cdot \underbrace{Bx_j}_{\text{eigenvector}}$$

$$= x_i \cdot \lambda_j x_j = \lambda_j \, x_i \cdot x_j = \begin{cases} \lambda_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Note that if $i = j$ then

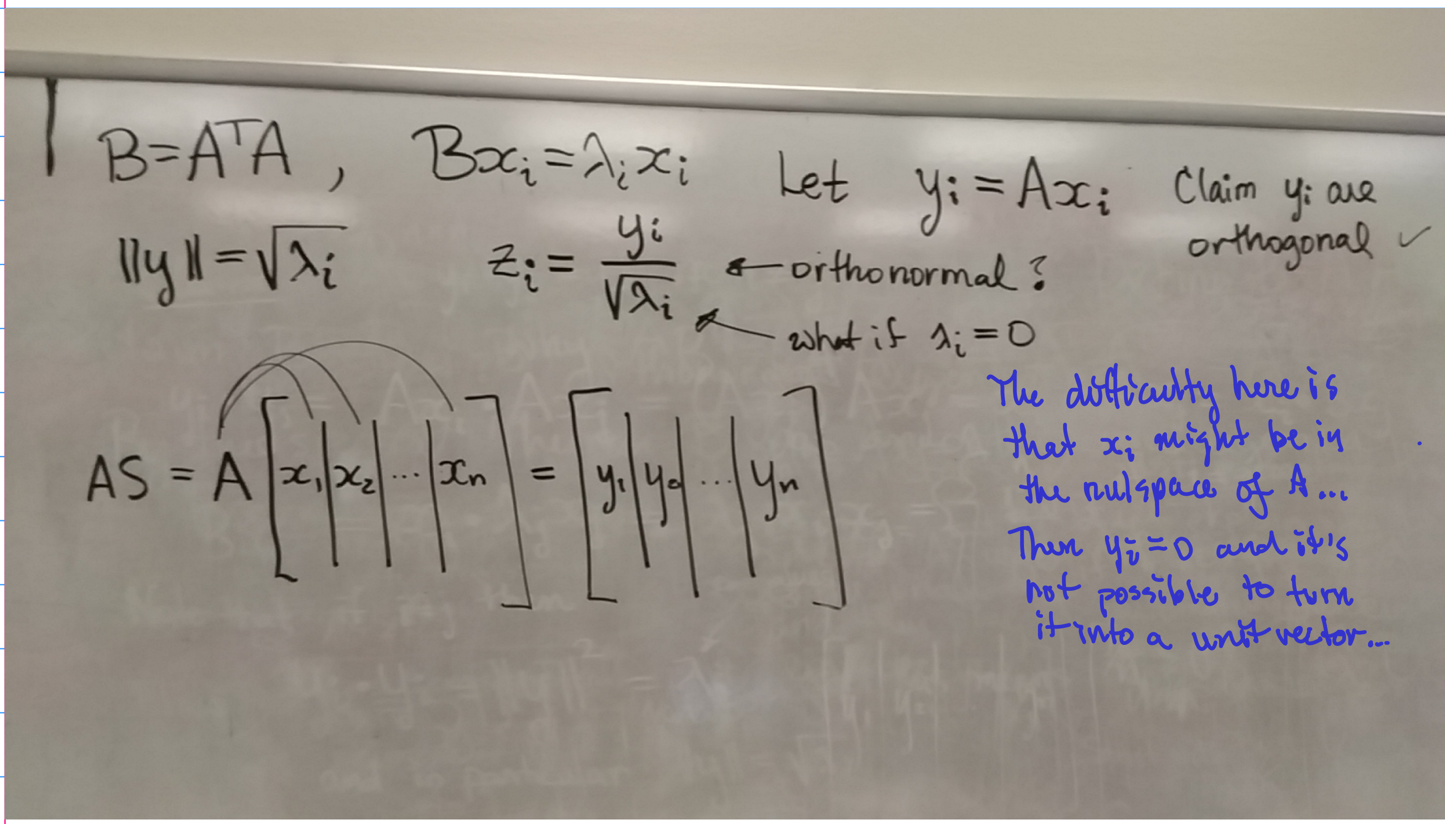
$$y_i \cdot y_i = \|y\|^2 = \underbrace{\lambda_i}_{\text{eigenvalues of B}}$$ and this means $\lambda_i \geq 0$.

note λ_i might be zero what would that mean about the original matrix A?

and in particular $\|y\| = \sqrt{\lambda_i}$

These are called the singular values of A.

norms are always non-negative...



Next time we'll work around the possibility that $\lambda_i = 0$ for some values...

Note if A is invertible then $B = A^TA$ is also invertible and none of the eigenvalues of B are zero and so none of the singular values of A are zero...