

The lecture was disrupted by the projector not working and then by the projector repair people ... Here are photos of the whiteboard that I made with my mobil phone... It seems some are blurry. I'll fill in the details here as I post things...

Chapter 7.1

Spectral Theorem: If $A \in \mathbb{R}^{n \times n}$ with $A^T = A$ hypothesis
Then A has an orthonormal conclusion orthonormal eigenbasis

We start with a review of the hypothesis and conclusion of the spectral theorem from before the holiday...

The hypothesis $A \in \mathbb{R}^{n \times n}$ and $A^T = A$ are

- ① easy to check
- ② occur naturally in some applications
- ③ when $A \neq A^T$ sometime a related matrix that is symmetric can be used instead.

That's the idea of the Singular Value Decomposition that is the focus of this last week...

What's the conclusion of the spectral theorem and why is it useful ..

Conclusion

Why important?

orthonormal eigenbasis
These words mean 4 things:

- ① unit vectors ✓ just normalize them
- ② basis □ ← didn't explain why but it's true when $A=A^T$.
- ③ orthogonal ← automatic when eigenvalues are different. When they are the same use Gram-Schmidt
- ④ eigenvector ← Start with these...

So you can diagonalize the matrix with a similarity transform $A=SDS^{-1}$

Makes S an orthogonal matrix means $S^T=S^{-1}$

Note that we proved everything except that $A^T=A$ implies there are exactly n linearly independent eigenvectors. ✓

This result can be obtained by an induction argument based on a reduction that works because A is symmetric. It's not difficult, but not the focus of this class, so we skip it. Instead we move to applications of the Spectral Theorem...

Two things left to do:

- ① Work a numerical example where $A=SDS^T$ to see how things work...
- ② Discuss the Singular Value Decomposition, which is an application of the spectral theorem to matrices which are not symmetric.

Since the projector is broken we start with the singular value decomposition, as that involves less writing...

Sorry, this slide turned out blurry too..

Chapter 7.4

Singular Value decomposition

Given $A \in \mathbb{R}^{m \times n}$ but not symmetric. Define $B = A^T A$
 Claim $B = B^T$. Why $B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$

By the spectral theorem B has an orthonormal eigenbasis.

$$Bx_i = \lambda_i x_i \quad \text{and} \quad x_i \cdot x_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

for $i, j = 1, \dots, n$

Then $B = SDS^T$ where $S = \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_n \\ | & | & | \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

Since B is symmetric the spectral theorem implies that S is an orthogonal matrix. Thus $S^{-1} = S^T$.

orthonormal columns

diagonal matrix

Idea to make a symmetric matrix out of one that is not symmetric.

$B = A^T A$, $Bx_i = \lambda_i x_i$ Let $y_i = Ax_i$ Claim y_i are orthogonal

Claim $y_i \cdot y_j = 0$ if $i \neq j$

$$y_i \cdot y_j = Ax_i \cdot Ax_j = (Ax_i)^T Ax_j = x_i^T A^T A x_j = x_i \cdot Bx_j$$

$$= x_i \cdot \lambda_j x_j = \lambda_j x_i \cdot x_j = \begin{cases} \lambda_j & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Note that if $i=j$ then

$$y_i \cdot y_i = \|y_i\|^2 = \lambda_i \quad \text{and this means } \lambda_i \geq 0.$$

and in particular $\|y_i\| = \sqrt{\lambda_i}$
 Norms are always non-negative...
 these are called the singular values of A .

eigenvectors of B

not necessarily unit vectors

eigenvector

note λ_i might be zero what would that mean about the original matrix A ?

$$B = A^T A, \quad Bx_i = \lambda_i x_i \quad \text{Let } y_i = Ax_i \quad \text{Claim } y_i \text{ are orthogonal } \checkmark$$

$$\|y_i\| = \sqrt{\lambda_i} \quad z_i = \frac{y_i}{\sqrt{\lambda_i}} \quad \leftarrow \text{orthonormal?}$$

$\leftarrow \text{what if } \lambda_i = 0$

$$AS = A \begin{bmatrix} | & | & \dots & | \\ x_1 & x_2 & \dots & x_n \\ | & | & \dots & | \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ y_1 & y_2 & \dots & y_n \\ | & | & \dots & | \end{bmatrix}$$

The difficulty here is that x_i might be in the nullspace of A ... Then $y_i = 0$ and it's not possible to turn it into a unit vector...

Next time we'll work around the possibility that $\lambda_i = 0$ for some values...

Note if A is invertible then $B = A^T A$ is also invertible and none of the eigenvalues of B are zero and so none of the singular values of A are zero...