the ith row and jth column of A. For instance, if

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

then A_{32} is obtained by crossing out row 3 and column 2,

$$\begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

so that

$$A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$(A_{32})_{31} = \begin{bmatrix} 5 & 0 \\ 4 & -1 \end{bmatrix}$$ 1st row

2nd col

$$((A_{32})_{31})_{12} = [4]$$

$$\det [4] = 4$$

TION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Generalization of formula

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \text{for some fixed } i$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det A = \sum_{j=1}^{2} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [4]$$

$$A_{12} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [3]$$

$\det A_{11} = 4$, $\det A_{12} = 3$

$$= (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12}$$

$$= a_{11} \det A_{11} - a_{12} \det A_{12} = 1 \cdot 4 - 2 \cdot 3 = -2$$

alternatively

$$\det A = \sum_{j=1}^{2} (-1)^{2+j} a_{2j} \det A_{2j}$$

$$A_{21} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [2] \qquad a_{21} = 3$$

$$A_{22} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [1] \qquad a_{22} = 4$$

Same, but different formula...

$$\det A = (-1)^{2+1} a_{21} \det A_{21} + (-1)^{2+2} a_{22} \det A_{22}$$

$$= -3 \cdot 2 + 4 \cdot 1 = -2$$

Verified in the 2×2 case the general formula

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$ for some fixed i

$_{ij}]$, the (i, j)-cofactor of A

$$\boxed{C_{ij} = (-1)^{i+j} \det A_{ij}}$$

later a cofactor matrix for Cramer's rule

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Determinant of an upper triangular matrix

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det A = \sum_{j=1}^{4} (-1)^{4+j} a_{4j} \det A_{4j}$$

$$= (-1)^{4+1} \underset{\to 0}{a_{41}} \det A_{41} + (-1)^{4+2} \underset{\to 0}{a_{42}} \det A_{42} + (-1)^{4+3} \underset{\to 0}{a_{43}} \det A_{43} + (-1)^{4+4} a_{44} \det A_{44}$$

$$\det A = 1 \cdot \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

by luck the entries in A_{44} are easy to relate to the original matrix

$$\det A = 1 \cdot 8 \cdot 1 \cdot 5$$

therefore

only $j=3$ term survives in the sum

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix} = \sum_{j=1}^{3} (-1)^{3+j} a_{3j} \det(A_{44})_{3j}$$

$$= (-1)^{3+3}\, 8 \det \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = 8(1\cdot 5 - 2\cdot 0) = 8\cdot 1\cdot 5$$

$$\det A = \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{1\cdot 8\cdot 5\cdot 1}_{\text{product along diagonal}}$$