Formula from: 285, 283

$$\boxed{\det A = a_{11}a_{22} - a_{12}a_{21}}$$

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

N

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

What's that mean?

Idea use recursion or induction to define something for all values of n.

How to make small matrices from bigger one

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Generalization:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

(same value for every i)

for any fixed i between 1 and n.

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

then A_{32} is obtained by crossing out row 3 and column 2,

$$\begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

so that

1st row

$$A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

2nd col

$$(A_{32})_{12} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$ 2nd row

2nd col

$$((A_{32})_{12})_{22} = [2].$$

Base case of the recursive definition 1x1 matrix

$$\boxed{\det[2] = 2.}$$

How does the definition

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

work for computing $\det A$?

Let suppose $A \in \mathbb{R}^{44}$

$$\det A = \sum_{j=1}^{4} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + (-1)^{1+3} a_{13} \det A_{13} + (-1)^{1+4} a_{14} \det A_{14}$$

Thus $\underbrace{\text{4 determinants of size } 3\times 3}$ are needed to find

1 determinant of size 4×4

$\det A_{11}$ where $A_{11} \in \mathbb{R}^{3\times 3}$

Also $\underbrace{\text{3 determinants of size } 2\times 2}$ are needed to find

1 determinant of size 3×3

Total # of terms in the sum is

$4 \cdot 3 \cdot 2 = 4!$

In general to compute determinant of a $n \times n$ matrix using the definition involves $n!$ number of terms...

<u>Theorem</u>: If $A = LU$ is the LU factorization of A where L is lower triangular with 1's on the diagonal and U is upper triangular (actually the row eschelon form of A). Then

$\det A$ = product of the terms on the diagonal of U.

$$\begin{array}{r} 24 \\ \times 6 \\ \hline 24 \\ 12 \\ \hline 144 \end{array}$$

$\det A = 2(-3)(4)(6) = -144$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -6 & 5 & 2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 4 & 8 \\ 0 & -3 & 6 & 2 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

where did these number come from

To find U we did row operations

$$\left\{\begin{array}{l} r_2 \leftarrow r_2 - \alpha_{21} r_1 \\ r_3 \leftarrow r_3 - \alpha_{31} r_1 \\ r_4 \leftarrow r_4 - \alpha_{41} r_1 \end{array}\right.$$

$$\left\{\begin{array}{l} r_3 \leftarrow r_3 - \alpha_{32} r_2 \\ r_4 \leftarrow r_4 - \alpha_{42} r_2 \end{array}\right.$$

$$\left\{ r_4 \leftarrow r_4 - \alpha_{43} r_3 \right.$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{bmatrix}$$

Then $LU = A$ where U is the row eschelonform of A made by the elimination steps

Suppose $A = \begin{bmatrix} 2 & 3 & 5 & 8 \\ 0 & -1 & 2 & 7 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

upper left

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

set $i=4$

$$= (-1)^{4+1} \underbrace{a_{41}}_{0} \det A_{41} + (-1)^{4+2} \underbrace{a_{42}}_{0} \det A_{42} + (-1)^{4+3} \underbrace{a_{43}}_{0} \det A_{43} + (-1)^{4+4} a_{44} \det A_{44}$$

$$\det A = 4 \det \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = 4 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Same as original indexing

$$\det \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = (-1)^{3+1} a_{31} \det (\quad) + (-1)^{3+2} a_{32} \det(-)$$

$$\boxed{+ (-1)^{3+3} a_{33} \det (-)}$$

only this survives

$$= 3 \det \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} = 3 \cdot 2 \cdot (-1)$$

Therefore

product along the diagonal

$$\det A = 4 \cdot (3 \cdot 2 \cdot -1) = -24$$