THEOREM 2

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

did this last time ✓

THEOREM 3

Row Operations

Let A be a square matrix.

Since we've been using row operations to make the LU factorization, it's useful to see how row operations affect the determinant.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Trying to understand the result from last time:

Theorem: If $A = LU$ is the LU factorization of A where L is lower triangular with 1's on the diagonal and U is upper triangular (actually the row eschelon form of A). Then

why? $\det A$ = product of the terms on the diagonal of U.

Page 188

$\det A = 2(-3)(4)(6) = -144$

understand by writing $A = LU$

$$\underbrace{\det A = \det(L)\det(U)}_{\text{Why is this?}} = 1 \cdot \det(U)$$

Goal is to understand why

$$\det(AB) = \det(A)\det(B)$$

when $A, B \in \mathbb{R}^{n \times n}$.

Since we can use row operations to construct any matrix, we study rows row operations first... anyway they are simpler...

M 3 **Row Operations**

$(i \neq j)$

Let A be a square matrix.

$r_i \leftarrow r_i - \alpha r_j$ → a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

$r_i \leftrightarrow r_j$ → b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

$r_i \leftarrow \alpha r_i$ → c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Examples (a) Elimination operation doesn't change the value of the determinant

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad r_2 \leftarrow r_2 - \alpha r_1 \qquad \begin{bmatrix} a & b \\ c - \alpha a & d - \alpha b \end{bmatrix}$$

$\det A = ad - bc$

$$\det \begin{bmatrix} a & b \\ c - \alpha a & d - \alpha b \end{bmatrix} = a(d - \alpha b) - b(c - \alpha a)$$

$$= ad - \cancel{\alpha ab} - bc + \cancel{\alpha ab}$$

$$= ad - bc.$$

Same determinant...

Example (b) Swapping rows changes the sign...

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad \det A = 2\cdot 5 - 3\cdot 4 = -2$$

sign changes

$r_1 \leftrightarrow r_2$

$$\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \qquad \det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} = 4\cdot 3 - 5\cdot 2 = 2$$

Example

(c) rescale a row

only one row

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad r_1 \leftarrow 3r_1 \qquad \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}$$

$$\det A = -2 \qquad \det \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix} = 6\cdot 5 - 9\cdot 4$$

$$= 30 - 36 = -6$$

whole determinant changed by 3

What about multiplying the entire matrix?

$$\det(3A) = \det\left(3\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = \det \begin{bmatrix} 6 & 9 \\ 12 & 15 \end{bmatrix} = 6\cdot 15 - 9\cdot 12$$

$$\left.\begin{array}{l} r_1 \leftarrow 3r \\ r_2 \leftarrow 3r \end{array}\right\} \text{Two row operations}$$

each changes the determinant by 3...

$\boxed{\text{same}}$

$$9 \det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 9(2\cdot 5 - 3\cdot 4) = -18$$

General rule

$$\boxed{\text{If } A \in \mathbb{R}^{n\times n} \text{ then } \det(\alpha A) = \alpha^{n} \det A}$$

(multiplies every row of A)

there are n rows in an $n\times n$ matrix.

We infer from the 2×2 case and the fact that determinant of an $n\times n$ matrix is defined recursively in terms of $(n-1)\times(n-1)$ matrices that the properties just demonstrated for 2×2 hold also for $n\times n$...

M 3

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Discussion of how much effort to compute det A using LU factorization compared to the definition.

Suppose $A \in \mathbb{R}^{5\times5}$. did elimination and got the row eschelon form

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & 2 & \vdots & \vdots & \vdots \\ 0 & 0 & -6 & \vdots & \vdots \\ 0 & 0 & 0 & 5 & \vdots \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

go in L matrix

Elimination steps to make those zeros...

$r_i \leftarrow r_i - \alpha_{ij} r_j$

Since the determinant doesn't change for each row operation then $\det A = \det U$.

How many $\underbrace{4+3+2+1}_{10}$ row operations.
each row involves 5 entries

Total of $10 \cdot 5 = 50$ terms... $\approx$ essentially $2n^3$ for general n

for large n the difference is even greater

to create U.

mult the diagonal 4 mult.

Total 54

Directly from the definition $\det(A)$ takes $5!$ operations...

$n!$ for general n

$$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 6\cdot 20 = 120$$

more than twice the work to use the definition when computing the determinant of a 5×5 matrix.

OREM 4 A square matrix A is invertible if and only if $\det A \neq 0$.

If $A \in \mathbb{R}^{n\times n}$ is not invertible then there is a row without a pivot → (this row makes the determinant zero)

~~Then~~ For example the row-eschelon form of A looks like

$$U = \begin{bmatrix} \square & ? & ? & ? & ? & ? \\ 0 & \square & ? & ? & ? & ? \\ 0 & 0 & 0 & \square & ? & ? \\ 0 & 0 & 0 & 0 & \square & ? \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\det U = 0}$$

whole row of zeros,

these zeros go here