$i \neq j$

M 3 **Row Operations**

Let A be a square matrix.

$r_i \leftarrow r_i - \alpha r_j$ → a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

$r_i \leftrightarrow r_j$ → b. If two rows of A are interchanged to produce B, then $\det B = \boxed{-}\det A$.

$r_i \leftarrow k r_i$ → c. If one row of A is multiplied by k to produce B, then $\boxed{\det B = \boxed{k} \cdot \det A}$.

Elimination step

(a) Matrix of $r_i \leftarrow r_i - \alpha r_j$ $\qquad B = [r_2 \leftarrow r_2 - 3r_1] A$

$n = 3\times3$

$$\det [r_2 \leftarrow r_2 - 3r_1] = \det \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1\cdot1\cdot1 = 1$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

by theorem

$$\det B = \det(\overset{E_1^{-1}}{[r_2 \leftarrow r_2 - 3r_1]} A) = \det A$$

$$= 1 \cdot \det A = \det(\overset{E_1^{-1}}{[r_2 \leftarrow r_2 - 3r_1]}) \det A$$

Row swap

(b) matrix of $r_i \leftrightarrow r_j$

$$\det [r_2 \leftrightarrow r_3] = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Use the theorem by taking $A = I$ and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det B = -\det A = -\det I = -1$$

$$\det I = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1\cdot1\cdot1 = 1$$

for any matrix A and $B = [r_2 \leftrightarrow r_3] A$ then

$$\det B = \det(\overset{E_i^{-1}}{[r_2 \leftrightarrow r_3]} A) = -\det A = \det \overset{E_i^{-1}}{[r_2 \leftrightarrow r_3]} \det A$$

Scaling operation

(c) $r_i \leftarrow k r_i$ Let $B = [r_1 \leftarrow 7r_1] A$

Theorem:

$$\boxed{\det B = 7 \det A}$$

$$[r_1 \leftarrow 7r_1] = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det[r_1 \leftarrow 7r_1] = \det \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 7 \cdot 1 \cdot 1 = 7$$

$$\det(\overset{E_1^{-1}}{[r_1 \leftarrow 7r_1]} A) = \det B = 7 \det A = \det \overset{E_1^{-1}}{[r_1 \leftarrow 7r_1]} \det A$$

This is enough to infer that

$$\det AB = \det A \det B \quad \text{for any } A, B \in \mathbb{R}^{n\times n}$$

If $\det A = 0$ what does that mean...

$\det U = 0$ where U is the row eschelon form of A.

this means there at least one zero on the diagonal of U. Which means that U is missing a pivot. So A is not invertible...

If A is not invertible, then AB is not invertible. Thus $\det(AB) = 0$.

Thus,

$$\boxed{\det AB = \det A \det B \quad \text{for any } A, B \in \mathbb{R}^{n \times n}}$$

is verified in the $\underbrace{0 = 0}$ case...

Suppose $\det AB \neq 0$...

This means the reduced row eschelon form of AB is just the identity matrix.

Note since we've already discussed $0 = 0$ case then $\det A \neq 0$ and $\det B \neq 0$

$$I = \underbrace{[r_i \leftarrow \alpha r_j] \cdots\cdots [r_i \leftrightarrow r_j][r_i \leftarrow \alpha r_i]}_{\substack{\text{guaussian elimination} \\ \text{written a matrix mult.}}} A$$

For convenience

$$I = E_n \cdots\cdots\cdots E_4 E_3 E_2 E_1 A$$

where each of the E_n's are an elementary row operation used in Gaussian elimination.

$$I = F_m \cdots\cdots\cdots F_4 F_3 F_2 F_1 B$$

Thus:

$$A = E_1^{-1} E_2^{-1} \cdots\cdots\cdots E_{n-1}^{-1} E_n^{-1}$$

$$\det A = (\det E_1^{-1})(\det E_2^{-1} \cdots\cdots E_{n-1}^{-1} E_n^{-1})$$

$$\det A = (\det E_1^{-1})(\det E_2^{-1}) \cdots\cdots (\det E_n^{-1})$$

$$\det B = (\det F_1^{-1})(\det F_4^{-1}) \cdots\cdots (\det F_m^{-1})$$

Thus

$$\det A \det B = (\det E_1^{-1})(\det E_2^{-1}) \cdots\cdots (\det E_n^{-1})$$

$$\times(\det F_1^{-1})(\det F_4^{-1}) \cdots\cdots (\det F_m^{-1})$$

$$= \det(AB)$$