The secret of determinants lies in how they change when row operations are performed. The following theorem generalizes the results of Exercises 19–24 in Section 3.1. The proof is at the end of this section.

M 3 **Row Operations**

Let A be a square matrix.

elimination a. If a multiple of one row of A is added to another row to produce a matrix B, then $\boxed{\det B = \det A.}$

swap b. If two rows of A are interchanged to produce B, then $\boxed{\det B = -\det A.}$

scaling c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

If $\det A = 0$ then the row echelon form of A call it U then also $\det U = 0$ this means there a zero on the diagonal which means a pivot is missing. So A is not invertible.

Trying to show $\boxed{\det AB = \det A \det B..}$

Suppose $\det AB = 0$. Then AB is not invertible.

Therefor either A or B or both are not invertible

so either $\det A = 0$ or $\det B = 0$ or both

Similarly if either $\det A = 0$ or $\det B = 0$ then AB is not invertible so $\det AB = 0$.

This explains the $0 = 0$ case for

$$\det AB = \det A \det B..$$

Suppose $\det A \neq 0$. Then A is invertible...

Then the reduced row eschelon form of A is just the identity matrix I.

$n=3$

matrix for $r_2 \leftarrow r_2 + 3r_1$	matrix for $r_2 \leftrightarrow r_3$	matrix for $r_3 \leftarrow 7r_3$
$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

Use there elementry matrices to factor the matrix A.

$$I = [r_1 \leftarrow \tfrac{1}{8} r_1] \cdots\cdots\cdots [r_2 \leftarrow r_2 - 2r_1]\, A$$

More simply

$$I = E_p \cdots\cdots\cdots E_3\, E_2\, E_1\, A$$

elementary row operations use to make the reduced row eschelon form of A.

Since the E_k's are invertible

$$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1} \cdots\cdots\cdots E_{p-1}^{-1} E_p^{-1}}_{\text{great big factorization of } A\ldots}$$

Meaning: any invertible matrix is a product of elementary row operations...

$$B = \underbrace{F_1^{-1} F_2^{-1} F_3^{-1} \cdots F_7^{-1}}_{}$$

prod. of elementary row operations...

Import point

these matrices are very simple...

elimination a. If a multiple of one row of A is added to another row to produce a matrix B, then $\boxed{\det B = \det A.}$

Example...

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

$$\boxed{\det \left(\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \right)} = \det B = 1 \cdot \det A = \boxed{\det \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \det A}$$

row swap

b. If two rows of A are interchanged to produce B, then $\boxed{\det B = -\det A.}$

$r_2 \leftrightarrow r_3$ by theorem

$$\det\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} = -1\det\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = -1$$

swapped rows of identity matrix

$$\boxed{\det\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}A} = \det B = -\det A = \boxed{\det\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\det A}$$

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

$k=7$

$r_3 \leftarrow 7r_3$

$$\det\begin{bmatrix}1&0&0\\0&1&0\\0&0&7\end{bmatrix} = 1\cdot1\cdot7 = 7$$

$$\boxed{\det\begin{bmatrix}1&0&0\\0&1&0\\0&0&7\end{bmatrix}A} = \det B = 7\det A = \boxed{\det\begin{bmatrix}1&0&0\\0&1&0\\0&0&7\end{bmatrix}\det A}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{p-1}^{-1} E_p^{-1}$$

$$\det A = \det E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{p-1}^{-1} E_p^{-1}$$

$$= (\det E_1^{-1})(\det E_2^{-1} E_3^{-1} \cdots E_{p-1}^{-1} E_p^{-1})$$

Thus,

$$\det A = (\det E_1^{-1})(\det E_2^{-1}) \cdots (\det E_p^{-1})$$

$$\det B = (\det F_1^{-1})(\det F_2^{-1}) \cdots (\det F_q^{-1})$$

So,

$$\det A \det B = (\det E_1^{-1}) \cdots (\det E_p^{-1})(\det F_1^{-1}) \cdots (\det F_q^{-1})$$

$$= (\det E_1^{-1}) \cdots (\det E_p^{-1})(\det F_1^{-1}) \cdots \underbrace{(\det F_{q-1}^{-1}) \cdot (\det F_q^{-1})}$$

$$= (\det E_1^{-1}) \cdots (\det E_p^{-1})(\det F_1^{-1}) \cdots \underbrace{(\det F_{q-1}^{-1} F_q^{-1})}$$

$$= \det (E_1^{-1} \cdots E_p^{-1} F_1^{-1} \cdots F_q^{-1}) = \det AB$$

Thus

$$\det AB = (\det A)(\det B)$$