(Jumping chapter 4, read on your own, let me know how it goes and we'll jump back as needed)

QR factorization

compare to LU factorization

lower triangular...

upper triangular...

Note assume $A \in \mathbb{R}^{m \times n}$ is square, but I want to assume the columns of A are linearly independent... Note $m \geq n$ (dim. ind. inequality).

$$\underset{m \times n}{A} = \underset{m \times n}{Q} \; \underset{n \times n}{R}$$

triangular matrix where I store stuff as I run an algorithm to create Q.

matrix with orthonormal columns...

$$\underset{m \times n}{Q} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

where $q_i \in \mathbb{R}^m$

$\|q_i\| = 1$ for $i = 1, \ldots, n$

$q_i \cdot q_j = 0$ for $i \neq j$

Idea... Gram-Schmidt algorithm...

$$A = \begin{bmatrix} a_1 \,\big|\, a_2 \,\big|\, \cdots \,\big|\, a_n \end{bmatrix} \quad \text{where } a_i \in \mathbb{R}^m$$

$t_1 = a_1$ $\qquad\qquad$ $q_1 = \dfrac{t_1}{\|t_1\|}$

$t_2 = a_2 - (q_1 \cdot a_2)\, q_1$

Claim t_2 is perpendicular to q_1

$$t_2 \cdot q_1 = (a_2 - (q_1 \cdot a_2)\, q_1) \cdot q_1$$

$$= a_2 \cdot q_1 - (q_1 \cdot a_2)\boxed{(q_1 \cdot q_1)}$$

unit vector

so $q_1 \cdot q_1 = 1$

$$= a_2 \cdot q_1 - q_1 \cdot a_2 = 0$$

$q_2 = \dfrac{t_2}{\|t_2\|}$

$t_3 = a_3 - (q_1 \cdot a_3)\, q_1 - (q_2 \cdot a_3)\, q_2$ $\qquad\qquad$ $q_3 = \dfrac{t_3}{\|t_3\|}$

$\vdots$

last step $\quad t_n = a_n - (q_1 \cdot a_n)\, q_1 - \cdots - (q_{n-1} \cdot a_n)\, q_{n-1}$ $\qquad\qquad$ $q_n = \dfrac{t_n}{\|t_n\|}$

$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\qquad$ $\|v\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$$\frac{v}{\|v\|} = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\sqrt{5}} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

Now

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

but what's R?

undoes the last step of the algorithm

$$R = \begin{bmatrix} \|t_1\| & q_1 \cdot a_2 & q_1 \cdot a_3 & \cdots & q_1 \cdot a_n \\ & \|t_2\| & \boxed{q_2 \cdot a_3} & & q_2 \cdot a_n \\ & 0 & & \ddots & \vdots \\ & & & & q_{n-1} \cdot a_n \\ & & & & \|t_n\| \end{bmatrix}$$